It is also possible to derive the joint cumulative distribution function (CDF) for ID correlated Rician RVs from the joint PDF in (4) as follows. First, we substitute the series expansion for $I_n(x)$ given by

$$I_n(x) = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \left( \frac{x}{2} \right)^{n+2k}$$  \hspace{1cm} (6)

for each of the three Bessel functions in (4). Then, the joint CDF of $R_1$ and $R_2$ becomes

$$P_{R_1,R_2}(r_1, r_2) = \frac{\exp \left( -\frac{\rho^2}{1+\rho^2} r_1^2 \right)}{\sigma^4 (1 - \rho^2)} \times \sum_{k, l_1, l_2, l_3=0}^{\infty} \frac{\rho^{k+2l_1+2l_2+2l_3}}{(1+\rho^2)^{k+l_1+l_2+l_3}} \times \gamma(k+l_1+l_2+1, \frac{r_1^2}{2(1-\rho^2)\sigma^2}) \times \gamma(k+l_2+l_3+1, \frac{r_2^2}{2(1-\rho^2)\sigma^2}).$$ \hspace{1cm} (7)

However, integrals of the form

$$I(r, z) = \int_0^r y^{2n+1} \exp(-2yz^2) dy$$ \hspace{1cm} (8)

can be evaluated in terms of the incomplete gamma function defined by

$$\gamma(p, u) = \int_0^u t^{p-1} \exp(-t) dt$$ \hspace{1cm} (9)

by making the substitution $t = 2yz^2$. When this is done in (8), we get

$$I(r, z) = \frac{1}{2z^{n+1}} \gamma(n+1, zr^2).$$ \hspace{1cm} (10)

Finally, applying (10) to (7) and simplifying gives the desired joint CDF as

$$P_{R_1,R_2}(r_1, r_2) = (1 - \rho^2) \exp \left( -\frac{\rho^2}{1+\rho^2} \right) \times \sum_{k, l_1, l_2, l_3=0}^{\infty} \frac{\rho^{k+2l_1+2l_2+2l_3}}{(1+\rho^2)^{k+l_1+l_2+l_3}} \times \gamma(k+l_1+l_2+1, \frac{r_1^2}{2(1-\rho^2)\sigma^2}) \times \gamma(k+l_2+l_3+1, \frac{r_2^2}{2(1-\rho^2)\sigma^2}).$$ \hspace{1cm} (11)

Authors’ Reply to Comments

George K. Karagiannidis and Dimitris A. Zogas

We appreciate greatly the comments of Dr. Simon [4] on our paper. Below is our reply to his comments:

1) (a) We were neither aware nor informed during the review process about the works related to the infinite-series representation (ISR) for the bivariate Rician pdf, included in [1] and [2]. This was the reason that we made the statement, “only the bivariate Rician pdf . . . in the past” in the Introduction of [3].

(b) We had no intention to avoid mentioning (and citing) these pioneering works, since in [3], an alternative approach (to that in [4]) was used to extract an ISR for the pdf, starting from a previously known integral representation (used also in [5]), and reaching a final expression which is apparently different from [4, eq. (4)].

2) We recognise the usefulness of [4, eq. (3)], and we agree that it is more general than [3, eqs. (6) or (9)], since the latter (ID case) is a special case of the former one. However, in [4, p. 1], the author claims that the presented expression for the ID case [4, eq. (4)] is “considerably simpler” than [3, eq. (6) or (9)]. We support that it is not correct, because of the following.

(a) In [4, p. 1], it is stated that, “In particular, although the result in [1, eq. (6)] has the appearance of a single infinite sum . . . and thus, is, in effect, a quadruple sum”. This is not exact, because in this case, the number of terms that need to be evaluated is $(i+1)(i+2)/2$ for each $i$, or totally

$$S = \sum_{i=1}^{N} \frac{(i+1)(i+2)}{2} = \frac{11N^2 + N^3}{6},$$

with $N$ being the truncation point. This number behaves as $N^3/6$ for large values of $N$, and is evidently much smaller than the total number of terms $N^4$, involved in a brute-force (uninformal) nested quadruple sum.

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The authors are with the Department of Electrical and Computer Engineering, Aristotle University of Thessaloniki, Thessaloniki 54 124, Greece (e-mail: geokarag@auth.gr; geokarag@ ieee.org; zogas@space.noa.gr). Digital Object Identifier 10.1109/TCOMM.2006.878805
(b) It is fair to compare the complexity between sums, when the summands are elementary functions (and not special as Bessel or hypergeometrics).

(i) The ISR [4, eq. (4)] contains a product of three Bessel functions, which is, in fact, a nested triple infinite sum, leading to a final expression which is a brute-force nested quadruple infinite sum with summands consisted from elementary functions. The total number of terms that need to be evaluated in such a sum is $N^4$.

(ii) The corresponding total number of terms in [3, eq. (9)] is

$$Z = N \times S = \frac{11N^2}{6} + N^3 + \frac{N^4}{6}$$

which behaves as $N^4/6$ for medium and large values of $N$. For example, when $N = 30$, then $N^4 = 810000$, while $Z = 133000$. Note that the Gamma function $\Gamma(x)$ involved in [3, eq. (9)] can be written in terms of elementary functions as $\Gamma(n) = (n - 1)!$ and $\Gamma(n/2) = 2^{1-n} \sqrt{\pi} (n-1)! / ((n-1)/2)!$.

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