Performance Analysis of Precoded Wireless OFDM With Carrier Frequency Offset

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Abstract—Precoding is proved to be highly efficient to improve the bit error rate (BER) of orthogonal frequency division multiplexing (OFDM) in frequency-selective fading channels. Nevertheless, most of the work reported in the literature ignores the radio frequency impairments such as carrier frequency offset (CFO), which is known to be of significant impact on OFDM. Consequently, this paper considers the performance evaluation of precoded OFDM (P-OFDM) in the presence of CFO. The performance of the considered P-OFDM is evaluated in terms of the signal-to-interference plus-noise-ratio (SINR) and BER. Various channel models are considered and closed-form analytical expressions are derived for the exact SINR. The obtained analytical results, corroborated by simulation, show that P-OFDM is substantially more sensitive to CFO compared with conventional OFDM. Generally speaking, if the normalized CFO is about 18%, then OFDM will outperform P-OFDM for most practical channel models. It is also interesting to note that the subcarriers in P-OFDM may experience different SINRs, which is not the case for conventional OFDM. This paper also considers the SINR of OFDM after the equalization process. The obtained results show that ignoring the impact of the equalization process results in inflated SINR degradation.

Index Terms—Frequency offset, Haar, interference, minimum mean square error (MMSE), orthogonal frequency division multiplexing (OFDM), precoding, synchronization, Walsh–Hadamard, Zadoff–Chu.

I. INTRODUCTION

Orthogonal frequency division multiplexing (OFDM) has received considerable research attention in wireless communications because of its ability to improve the spectral efficiency and immunity to inter-symbol interference (ISI) in frequency-selective multipath fading environments. Furthermore, the ISI can be avoided without the need for complicated equalizers, with the aid of sufficient cyclic prefix (CP), or using interference cancellation schemes [1]. Owing to the remarkable advantages of OFDM, it has been adopted for several wireless communication standards such as digital video broadcasting [2] and digital audio broadcasting [3]. Additionally, OFDM is used as the transmission scheme in the physical layer for microwave access (WiMAX) [4], long-term evolution (LTE) standards [5], [6], optical wireless communications (OWCs) [7], [8], and it is considered as a promising candidate for the fifth-generation standards [9]. The channel is typically modeled as frequency-selective for WiMax and LTE, flat for OWC in the presence of atmospheric turbulences, and additive white Gaussian noise (AWGN) for OWC without atmospheric turbulence.

However, OFDM has critical drawbacks such as sensitivity to synchronization errors due to radio frequency (RF) front-end related imperfections, which are often inevitable due to channel variation, components mismatch, and manufacturing defects. Common examples are carrier frequency offsets (CFO) [10], [11], timing offsets [12], and I/Q imbalance [13], [14]. Generally speaking, RF impairments cause loss of orthogonality between subcarriers, and hence, inter-carrier interference (ICI) is introduced. OFDM also suffers from large peak-to-average power ratio (PAPR) problem [15].

Despite the many advantages of OFDM, its bit error rate (BER) performance is similar to single-carrier (SC) systems in fading channels [16]. Therefore, incorporating diversity techniques is essential to improve the performance of OFDM in frequency-selective wireless channels. Moreover, reducing the PAPR should be considered to avoid BER degradation due to the power amplifier nonlinearity. In this context, using precoding with OFDM provides considerable diversity gain and robustness against the frequency selectivity of the channel, and can reduce the PAPR [17]. Particularly, precoded-OFDM (P-OFDM) based on unitary transforms, such as Walsh–Hadamard transform (WHT), has received significant attention from the researchers and is considered as energy and spectrally efficient approach that can drastically enhance the performance of wireless systems [17]–[19]. Other transforms such as the Haar transform [20] and Zadoff–Chu matrix transform (ZCMT) have been considered in the literature as well [21]. More recently, Popescu and Popescu [22] demonstrated that using sub-band P-OFDM may enhance the immunity of OFDM to narrowband interference. Furthermore, the performance of P-OFDM over power-line communications with impulse noise was considered in [23] by using various equalization techniques. The presented results show that P-OFDM outperforms conventional OFDM in such scenarios.
A. Prior Work

Estimating the CFO and investigating its effect on the performance of OFDM-based wireless communication systems has motivated a vast amount of research [24]–[44]. In most previous research, it is common to measure the performance degradation due to CFO in terms of the average signal-to-interference plus-noise-ratio (SINR) [24]–[29]. A lower and upper bounds on the average SINR in the presence of CFO in multipath fading channels are derived in [24] and [25]. The derivation of an exact expression for the average SINR requires $N$-fold numerical integrations to average out the $N$ correlated random variables, where $N$ is the number of subcarriers. However, using an indirect mathematical analysis, the exact expression for the average SINR is derived by Hamdi [29] in the form of a single integral. The impact of CFO and phase noise (PN) is investigated in [31] and [32], where an exact signal-to-interference ratio expression for OFDM in the presence of CFO and PN in double-selective fading channels is presented in [32].

The analysis of BER deterioration due to CFO is considered widely in the literature [33]–[37]. Approximated BER of OFDM in the presence of CFO for AWGN and multipath fading channels is presented in [33] and [34]. BER analysis of binary phase shift keying (BPSK) and OFDM with random residual frequency offset has been considered in [35] where a closed-form BER expressions for BPSK in AWGN, flat, and selective fading channels are derived. Furthermore, the authors provided expressions for quadrature phase shift keying (QPSK), but only for AWGN and flat fading channels. However, the expressions derived in [35] are valid only for small normalized CFO. Therefore, Mahesh and Chaturvedi [36] extended the results of [35] for the BPSK case to be valid for any normalized CFO. Furthermore, a closed-form expression for the symbol error rate (SER) of QPSK-OFDM with a frequency offset in Rayleigh selective fading channels is derived in [37]. In addition, the SER of QPSK-OFDM with frequency offset in flat Rayleigh fading channels that is valid for any frequency offset was provided. A formula for the spectral efficiency of OFDM systems in a frequency-selective multipath fading channel, in the presence of CFO, is derived in [38], whereas the impact of the CFO on the spectrum sensing of OFDM signals is investigated in [39]. Likewise, the effect of CFO on different types of SC frequency division multiple access (FDMA) systems is demonstrated in [40], [41], whereas an analytical expression for the error vector magnitude measure of SC-FDMA system under CFO and joint transmit–receive PN is derived in [42]. The performance degradation due to CFO in virtual multiple-input single-output systems is highlighted in [43]. The impact of CFO on multiuser detection and estimation performance of LTE system is demonstrated in [44]. Finally, a concatenated P-OFDM system is proposed in [30] to enable channel estimation and mitigate the effects of ICI caused by CFO.

B. Contribution

As can be noted from the aforementioned discussion, the impact of CFO on OFDM has been considered extensively in the literature. However, to the best of the authors’ knowledge, no analysis has been reported in the open technical literature on the impact of CFO on P-OFDM wireless systems. Consequently, this paper considers the performance evaluation of P-OFDM systems in the presence of CFO where WHT is used as the precoding transform due to its implementation efficiency [19]. In particular, the paper focuses on analyzing the average SINR of P-OFDM in AWGN, flat, and frequency-selective fading channels. Moreover, the conventional equalized OFDM case is considered as well. The derived average SINR expressions are corroborated by Monte Carlo simulation results. The obtained analytical and simulation results show that P-OFDM is significantly more sensitive to CFO compared with conventional OFDM, which may dilute the P-OFDM transmit diversity advantage, or even make its performance worse than the conventional one. Moreover, P-OFDM has entirely different behavior as compared with OFDM where the SINR can be different for each subcarrier. Therefore, while certain subcarriers are lightly affected by CFO, other subcarriers may experience severe SINR degradation, which may drive the overall BER of P-OFDM to become worse than conventional OFDM. Note that the performance of P-OFDM is also considered with the Haar and ZCMTs.

C. Structure and Notations

The rest of the paper is organized as follows. Section II presents P-OFDM system and channel models in the presence of CFO. In Section III, the average SINR expressions of P-OFDM in AWGN, flat, and frequency-selective Rayleigh fading channels in the presence of CFO are derived. Numerical and simulation results with discussions are provided in Section IV, whereas conclusions and closing remarks are presented in Section V.

Notations: Unless otherwise stated, lower and upper case bold letters such as $\mathbf{x}$ and $\mathbf{X}$ denote vectors and matrices, respectively. The matrices $\mathbf{I}$ and $\mathbf{F}$ denote the identity matrix and the discrete Fourier transform (DFT) matrix, respectively. The operators $\mathbf{P}$, $\mathbf{H}$, $\mathbf{F}$, $\mathbf{P}^*$, $\mathbf{H}^*$, and $\mathbf{P}^{-1}$ denote the transpose, complex-conjugate, transpose, and matrix inverse operations, respectively. The operators $\mathbb{E} \left[ \cdot \right]$ and $| \cdot |$ denote the statistical expectation and the absolute value, respectively.

II. P-OFDM System and Channel Models

In P-OFDM, a sequence of complex data symbols $\mathbf{a} = [a_0, a_1, \ldots, a_{N-1}]^T$, each of which has an average power $P_a$ is applied to $N$-point WHT to generate the precoded data sequence

$$\mathbf{b} = \mathbf{Wa}$$

(1)

where $\mathbf{W}$ is the normalized $N$-point Walsh–Hadamard matrix. The columns and rows of $\mathbf{W}$ are normalized to a unit norm, such that all its elements are equal to $\pm 1/\sqrt{N}$ and $\mathbf{W}^{-1} \mathbf{W} = \mathbf{I}$. The $m$th sample, $m \in \{0, 1, \ldots, N - 1\}$, at the output of WHT block is a linear combination of all data symbols, and can be expressed as [19]

$$b_m = \mathbf{w}_m^T \mathbf{a} = \sum_{n=0}^{N-1} W_{m,n} a_n$$

(2)
where $w_m$ is the $m$th row of $W$

$$w_m = [W_{m,0}, W_{m,1}, \ldots, W_{m,N-1}]$$

and $W_{m,n}$ is the element of the $m$th row and $n$th column of $W$, which can be obtained as

$$W_{m,n} = \frac{1}{\sqrt{N}} (-1)^{\frac{\log_2(N)-1}{m-n}}$$

with $m_r$ and $n_r$ are the bit representation of the integer values $m$ and $n$, respectively.

The composite symbols vector $b$ is applied to an $N$-point inverse DFT to generate the time domain samples, similar to conventional OFDM, thus

$$x = F^H b$$

where $F^H$ is the Hermitian transpose of the normalized $N \times N$ DFT matrix $F$. The elements of $F^H$ are defined as $F_{i,n}^H = \sqrt{N}e^{j2\pi in/N}$ where $i$ and $n$ denote the row and column numbers \{i, n\} ∈ 0, 1, ..., N - 1, respectively, and $N \triangleq 1/N$. The $i$th sample of $x$ can be obtained as

$$x_i = \sqrt{N} \sum_{n=0}^{N-1} b_n e^{j2\pi inN}, i = 0, 1, \ldots, N - 1.$$  \hspace{1cm} (6)

In the case of imperfect frequency synchronization, the time-domain received sequence $y = [y_0, \ldots, y_{N-1}]^T$, after discarding the $P$ CP samples, can be expressed as [10], [11]

$$y = C(\varepsilon)\tilde{H}F^H b + z$$

where $\varepsilon \triangleq \frac{\Delta f}{f_0} \in (-0.5, 0.5)$ is the normalized CFO, $\Delta f$ is the actual CFO, $R_s$ is the data symbol rate, and $C(\varepsilon)$ represents the accumulated phase shift on the time-domain samples caused by the normalized CFO. Thus

$$C(\varepsilon) = \text{diag}\left(\left[e^{j2\pi\varepsilon N/0}, e^{j2\pi\varepsilon N/1}, \ldots, e^{j2\pi\varepsilon N/(N-1)}\right]\right).$$

(8)

Given that the channel has $L_h + 1$ multipath components, the channel matrix $H$ is an $N \times N$ circulant matrix with $h_0$ on the principal diagonal and $h_1$, ..., $h_L$ on the minor diagonals, $z = [z_0, z_1, \ldots, z_{N-1}]^T$ denotes the system noise where $z_i \sim \mathcal{CN}(0, \sigma_z^2)$. The sequence $y$ is applied to the DFT that produces the sequence $r = [r_0, r_1, \ldots, r_{N-1}]^T$, where

$$r = FC(\varepsilon)\tilde{H}F^H b + Fz$$

(9)

which can be written as

$$r = \mathcal{H}b + \eta.$$  \hspace{1cm} (10)

The $m$th element of $r$ can be expressed as

$$r_m = \sum_{p=0}^{N-1} \mathcal{H}_{m,p} b_p + \eta_m$$

(11)

where $\mathcal{H}_{m,p}$ is the element of the $m$th row and $p$th column of $\mathcal{H}$, which can be expressed as

$$\mathcal{H}_{m,p} = \sqrt{N} \sum_{k=0}^{N-1} \sum_{l=0}^{L_h} h_l e^{j2\pi k(p-m+\varepsilon)N} e^{-j2\pi l p N}$$

(12)

or, after some algebraic manipulations

$$\mathcal{H}_{m,p} = \alpha_{m,p} H_p$$

(13)

where $H_p = \sum_{l=0}^{L_h} h_l e^{-j2\pi lpN}$ denotes the channel frequency response at subcarrier $p$ and

$$\alpha_{m,p} = \frac{\sin(\pi(p-m+\varepsilon))}{N \sin(\pi(p-m+\varepsilon))} e^{j\pi(1-N)(p-m+\varepsilon)}.$$ \hspace{1cm} (14)

The diagonal elements of $\mathcal{H}$ can be computed by setting $m = p$, which gives

$$\alpha_{m,m} \triangleq \alpha = \frac{\sin(\pi\varepsilon)}{N \sin(\pi\varepsilon)} e^{j\pi(1-N)}.$$ \hspace{1cm} (15)

To extract the vector $b$ from (10) without ICI, the DFT output vector $r$ should be equalized using the inverse of the channel matrix $\mathcal{H}$, i.e., $\mathcal{H}^{-1}$. However, such process is prohibitively expensive due to the complexity of estimating $\mathcal{H}$ as well as the complexity of the matrix inversion process. By noting that the diagonal elements of $\mathcal{H}$ are dominant as compared with the off-diagonal elements [45], then, it is more feasible to estimate the diagonal elements of $\mathcal{H}$ and use them to equalize $r$, which is the approach used in this paper. Therefore, the minimum mean square error (MMSE) equalizer output can be expressed as [46]

$$v = \mathcal{H}^* b + \tilde{\eta}$$

(16)

where $\tilde{\eta} = \tilde{H} \eta, \tilde{H} = \text{diag}\left\{[\tilde{H}_0, \tilde{H}_1, \ldots, \tilde{H}_{N-1}]\right\}$, and

$$\tilde{H}_m = \frac{H_{m,m}}{|H_{m,m}|^2 + \frac{1}{\text{SNR}}} = \frac{\alpha^* H_m}{|\alpha H_m|^2 + \frac{1}{\text{SNR}}}.$$ \hspace{1cm} (17)

In (17), the SNR $= \frac{P_s}{\sigma_z^2}$. As can be noted from (17), the CFO affects the equalization process, and accurate channel estimates require the knowledge of $\varepsilon$, which might not be available at the receiver. Nevertheless, the sensitivity of the system to accurate knowledge of $\alpha$ is mostly negligible because $\alpha \approx 1$ for $\varepsilon \leq 0.2$. Moreover, at moderate and high SNR values, the denominator of (17) is dominated by $1/$SNR.

The $m$th element of $v, m \in \{0, 1, \ldots, N - 1\}$, is given by

$$v_m = \sum_{p=0}^{N-1} \tilde{H}_m \mathcal{H}_{m,p} b_p + \tilde{\eta}_m$$

(18)

where $\tilde{\eta}_m = \tilde{H} \eta_m$. After equalization, the inverse WHT (iWHT) is computed to produce the sequence $d = W^{-1}v = W^{-1}(\mathcal{H}b + \tilde{\eta})$, where the $k$th element of $d$ is given by

$$d_k = w_k^{-1}v_k = \sum_{m=0}^{N-1} W_{k,m}^{-1} v_m = \sum_{m=0}^{N-1} \sum_{p=0}^{N-1} W_{k,m}^{-1} \mathcal{H}_{m,p} b_p + \zeta_k$$

(19)
where \( w_k^j \) is the \( k \)-th row of \( W^{-1} \) and \( \zeta_k = \sum_{m=0}^{N-1} W^{-1}_{k,m} \hat{h}_m \).

By extracting the term that corresponds to the \( k \)-th data element, then

\[
d_k = \beta_{k,k} a_k + \sum_{n=0}^{N-1} \beta_{k,n} a_n + \zeta_k
\]

(20)

where

\[
\beta_{k,n} = \sum_{m=0}^{N-1} \sum_{p=0}^{N-1} W^{-1}_{k,m} W_{p,n} \hat{H}_m v_{m,p}.
\]

(21)

### III. SINR ANALYSIS OF P-OFDM

Based on (20), the average SINR for the \( k \)-th subcarrier can be defined as

\[
\text{SINR}_k = \frac{\mathbb{E} \left[ |\beta_{k,k} a_k|^2 \right]}{\mathbb{E} \left[ \sum_{n=0}^{N-1} \beta_{k,n} a_n / n \neq k \right]^2 + \sigma_k^2}
\]

(22)

which can be simplified to

\[
\text{SINR}_k = \frac{\mathcal{P}_s \mathbb{E}(|\beta_{k,k}|^2)}{\mathcal{P}_s \sum_{n=0}^{N-1} \sum_{n \neq k} \mathbb{E}(|\beta_{k,n}|^2) + \sigma_k^2}
\]

(23)

where \( \sigma_k^2 = \mathbb{E} \left[ \sum_{m=0}^{N-1} |\hat{H}_m|^2 \right] \). Substituting (21) in (23) gives the \( \text{SINR}_k \) in (24) at the bottom of the page where

\[
\Psi^1_{k,m,p,v,i} = W_{k,m} W_{k,p} W_{i,k}, \quad \Psi^2_{k,m,p_v,p_i} = W_{k,m} W_{p,n} W_{k,n} W_{v,i} \quad \text{and} \quad u = \frac{1}{\text{SNR}}.
\]

As can be noted from (24), evaluating \( \text{SINR}_k \) for an arbitrary \( H \) is infeasible because computing the expected values in the numerator and denominator of (24) requires averaging over the joint probability density function (PDF) of the channel coefficients \( H_m, m = [0, 1, \ldots, N-1] \), which requires using \( N \)-fold integrals to average out the \( N \) correlated random variables for each subcarrier [29], which is prohibitively expensive. Therefore, (24) can be used to compute \( \text{SINR}_k \) semi-analytically by generating large number of realizations of \( H \), and then computing

\[
\mathbb{E} \left[ \sum_{v,i,m,p} \Psi^1_{k,m,p,v,i} \alpha_{m,p} \rho_{v,i} \mathbb{E} \left[ \frac{H_{m} H_{p}}{|H_{m}|^2 + u} \cdot \frac{H_{i} H_{v}}{|H_{i}|^2 + u} \right] \right]
\]

(24)

where \( \alpha^* = \frac{\alpha^*}{|\alpha|^2 + \frac{\alpha^*}{\text{SNR}}} \) and \( \rho^i \) respectively, which implies that \( \beta_{0,0} = |\beta_{1,1}| \).

Therefore, \( \text{SINR}_0 = \text{SINR}_1 = \text{SNR} \). It is worth noting that simplifying the values \( \beta_{0,0} \) and \( \beta_{1,1} \) is based on the fact that the elements of the first row and column of \( W \) and \( W^{-1} \) are all ones, and hence, \( W_{0,m} W_{0,p} = \frac{1}{N}, \sum_{m=0}^{N-1} \sum_{p=0}^{N-1} \alpha_{m,p} = N \forall \varepsilon \), and \( \sum_{m=0}^{N-1} \sum_{p=0}^{N-1} W_{k,m} W_{p,n} \alpha_{m,p} = \sigma_{\epsilon}^2 \).

As another example, consider subcarrier \( d_2 \), which can be expressed as

\[
d_2 = \beta_{2,2} a_2 + \beta_{2,3} a_3 + \zeta_2.
\]

(30)

Although it is intractable to obtain a closed-form expression for the \( \text{SINR}_k \) in frequency-selective channels, it is shown in the following two sections that closed-form expressions can be derived for the special cases of AWGN and flat fading channels.

### A. AWGN Channels

For AWGN channels, \( H_{p} = 1 \forall p \). Consequently, (24) is reduced to

\[
\text{SINR}_k = \frac{\sum_{m=0}^{N-1} \sum_{p=0}^{N-1} W^{-1}_{k,m} W_{p,k} \alpha_{m,p}^2}{\sum_{m=0}^{N-1} \sum_{p=0}^{N-1} W^{-1}_{k,m} W_{p,k} \alpha_{m,p}^2 + \frac{1}{\text{SNR}}}.
\]

(26)

Moreover, the output of the IWHT at the receiver side can be expressed as

\[
d = WHF \left( C(\varepsilon) HF \left( Wa + z \right) \right) = Ma + \zeta.
\]

(27)

where \( M \) is given by (28) at the bottom of the next page, \( \zeta = [\zeta_0, \zeta_1, \ldots, \zeta_{N-1}]^T \) where \( \zeta_k = \frac{\alpha^*}{|\alpha|^2 + \frac{\alpha^*}{\text{SNR}}} \sum_{m=0}^{N-1} W^{-1}_{k,m} \hat{h}_m \) and

\[
\mathbb{E} \left[ |\zeta_k|^2 \right] \triangleq \sigma_{\zeta_k}^2
\]

can be expressed as

\[
\sigma_{\zeta_k}^2 = \left| \frac{\alpha^*}{|\alpha|^2 + \frac{\alpha^*}{\text{SNR}}} \right|^2 \sigma_z^2.
\]

(29)

By noting that \( M \) is a block matrix, then not all subcarriers contribute to the ICI. For example, subcarriers \( d_0 \) and \( d_1 \) can be expressed as \( d_0 = \beta_{0,0} a_0 + \zeta_0 \) and \( d_1 = \beta_{1,1} a_1 + \zeta_1 \), and, hence, \( d_0 \) and \( d_1 \) do not suffer from ICI. Moreover, the attenuation factors \( \beta_{0,0} \) and \( \beta_{1,1} \) can be written as \( \alpha^* / |\alpha|^2 + \frac{\alpha^*}{\text{SNR}} \) and \( \alpha^* \sigma_{\epsilon}^2 \), respectively, which implies that \( \beta_{0,0} = |\beta_{1,1}| \).
Therefore, only $a_3$ contributes to the ICI. Moreover, because $|β_{2,2}| = |β_{3,3}|$ and $|β_{2,3}| = |β_{3,2}|$, the SINRs of $d_2$ and $d_3$ are identical.

### B. Flat Fading Channels

In flat Rayleigh fading, all channel coefficients are equal, and hence, $H_k = H$ $∀k$, $H_k = H = \frac{\mathcal{W}_k\epsilon'}{\lambda_k}$ By defining $λ_1 = \frac{|H|^4}{|H|^2 + u}$ and $λ_2 = \frac{|H|^2}{|H|^2 + u}$, then (24) reduces to

\[
\text{SINR}_k = \left( \sum_{n=0}^{N-1} \sum_{p=0}^{N-1} |H_{k,m}|^2 \right)^2 \\
\sum_{n=0}^{N-1} \sum_{p=0}^{N-1} |H_{k,m}|^2 + \frac{1}{\text{SINR}} \frac{E[λ_1]}{E[λ_2]}.
\]

Assuming that $H \sim \mathcal{CN}(0, σ^2)$, then $|H|^2 \sim \theta \sim \chi^2$, and by considering that $σ^2 = 1$, the PDF of $θ$ is given by

\[
f_θ(θ) = e^{-θ}, \quad θ > 0.
\]

Consequently, the values of $E[λ_1]$ and $E[λ_2]$ can be evaluated as

\[
E[λ_1] = \int_0^∞ \frac{θ^2}{(θ + u)^2} e^{-θ} dθ = e^{-θ}.
\]

Using the change of variables $ρ = θ + u$, expanding the resultant terms, then $E[λ_1]$ can be rewritten as

\[
E[λ_1] = e^u \int_0^∞ e^{-ρ} u^2 e^{-ρ} dρ = e^u \int_0^∞ e^{-ρ} dρ.
\]

Using (36) and after some straightforward manipulations, the integral in (34) can be evaluated as

\[
E[λ_1] = (1 + u) + (2 + u)ue^u Ei(-u)
\]

where $Ei(·)$ is the exponential integral.

Similarly

\[
E[λ_2] = \int_0^∞ \frac{θ}{(θ + u)^2} e^{-θ} dθ.
\]

Using the change of variables $ρ = θ + u$, expanding the resultant terms and evaluating the integral gives

\[
E[λ_2] = \int_0^∞ e^{-ρ} \frac{e^{-ρ}}{ρ^2} dρ = \int_0^∞ e^{-ρ} (ρ - 1) dρ.
\]

Therefore, a closed-form expression for the SINR in flat Rayleigh fading channels can be obtained by substituting $E[λ_1]$ and $E[λ_2]$ in (31), which are given in (35) and (37), respectively.

It is also worth noting that the output structure of the IWHT at the receiver side is similar to the AWGN case, which can be written as depicted in (27), where

\[
β_{k,n} = \frac{α^2 H^2}{|α H|^2 + \frac{1}{\text{SNR}}} \sum_{m=0}^{N-1} W_{k,m}^* W_{n,m}^* \text{SNR}
\]

and $ζ_k = \frac{α^2 H^2}{|α H|^2 + \frac{1}{\text{SNR}}} \sum_{m=0}^{N-1} W_{k,m}^* η_m$ whose variance is given by

\[
σ^2 = \int_0^∞ \frac{α^2 H^2}{|α H|^2 + \frac{1}{\text{SNR}}} \text{SNR}
\]

As a special case, the first two diagonal elements in the matrix $M$ can be simplified to $β_{0,0} = \frac{α^2 |H|^2}{|α H|^2 + \frac{1}{\text{SNR}}}$ and $β_{1,1} = \frac{α^2 |H|^2}{|α H|^2 + \frac{1}{\text{SNR}}} e^{iπε}$. Therefore, similar to the AWGN case $\text{SINR}_0 = \text{SINR}_1$, where

\[
\text{SINR}_0 = \frac{E[β_{0,0}]^2}{σ^2 \text{SNR}} = \text{SNR} \frac{E[λ_2]}{E[λ_1]}.
\]

By noting the values of $E[λ_1]$ and $E[λ_2]$ given in (35) and (37), respectively, it can be numerically demonstrated that $E[λ_1] < 1$ for $\text{SNR} ≫ 0$ dB, which implies that $\text{SINR}_0[\text{Flat}] < \text{SINR}_0[\text{AWGN}]$.

It is also worth noting that SINR of the equalized conventional OFDM can be derived as a special case of the P-OFDM by replacing the WHT/IWHT matrices by the identity matrix, i.e.,

\[
M = \begin{bmatrix}
β_{0,0} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & β_{1,1} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & β_{2,2} & β_{2,3} & 0 & 0 & 0 & 0 \\
0 & 0 & β_{3,2} & β_{3,3} & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & β_{N-1, N-1} \\
0 & 0 & 0 & 0 & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & β_{N-1, N-1} \end{bmatrix}
\]
\( W_{n,m} = 1 \) for \( n = m \), and 0 otherwise. As shown in Appendix I, the SINR for the AWGN case is given by

\[
\text{SINR} = \frac{|\alpha|^2}{1 - |\alpha|^2 + \frac{1}{\text{SNR}}}. \tag{40}
\]

By comparing the SINR with and without MMSE equalization, which are given by (40) and [25, Eq. (20)], respectively, it can be concluded that both formulas are identical. Therefore, the equalization has no impact on the average SINR in the AWGN channel case.

For the flat Rayleigh fading, the SINR for equalized conventional OFDM can be expressed as

\[
\text{SINR} = \frac{|\alpha|^2}{1 - |\alpha|^2 + \frac{1}{\text{SNR}} \frac{\text{Ei}(-u)}{\text{Ei}(-u) + (u+1)\text{Ei}(-u) + \text{Ei}(-u)}}. \tag{41}
\]

As it can be noted from (40) and (41), the subcarrier index is dropped because SINR\(_m\) is the same for all values of \( m \). Moreover, both formulas are simplified using the fact that \( \sum_{p=0}^{N-1} |\alpha_{m,p}|^2 = 1 - |\alpha|^2 \). It is straightforward to show that when SNR \( \to \infty \), SINR for both the AWGN and flat fading converges to \( \frac{|\alpha|^2}{1 - |\alpha|^2} \). Furthermore, without equalization \( \mathbb{E}[\alpha_2] = \mathbb{E}[\alpha_1] \), the average SINR in (41) becomes equal to the SINR of the AWGN case.

Although the main focus of this paper is P-OFDM using WHT, applying the obtained results to any other precoding transform is straightforward, and can be achieved by replacing the transform matrix \( W \) by the desired transform matrix, and then use (24) to compute \( \text{SINR}_k \), \( k = 0, 1, \ldots, N - 1 \).

IV. NUMERICAL RESULTS

This section presents the performance evaluation of P-OFDM systems in terms of SINR and BER over AWGN, flat Rayleigh fading, and frequency-selective Rayleigh fading channels. The OFDM symbol structure is generally based on the LTE [48] configuration except that the number of subcarriers is \( N = 64 \). The data part of the OFDM symbol has a duration of \( T_u = 66.68 \) \( \mu s \) and the subcarrier spacing is 15 kHz. The number of CP samples \( P = 16 \), with a duration of \( T_{cp} = 16.67 \) \( \mu s \). The data symbols are drawn from QPSK constellation and the total OFDM symbol period is \( T_i = 83.35 \) \( \mu s \).

The frequency-selective fading channel considered in this paper has five taps with normalized delays of \([0, 1, 3, 5, 10]\) samples and average taps’ gains of \([0.47, 0.29, 0.12, 0.07, 0.05]\). Therefore, the root mean-square delay spread of the channel \( \sigma(\tau) = 2.52 \) \( \mu s \) given that the sampling period is \( T_s/80 \approx 1.04 \) \( \mu s \). The multipath components’ gains are generated as complex independent Gaussian random variables. The channel gains remain constant for a given OFDM symbol period, but changes randomly over different symbols, which corresponds to a quasi-static channel. All the presented results are obtained using WHT unless it mentioned otherwise.

Fig. 1 depicts SINR\(_k\) versus the subcarriers’ indices for the OFDM and P-OFDM over AWGN, flat, and frequency-selective fading channels using \( \varepsilon = 0.1 \) and SNR = 20 dB. The analytical results correspond to the AWGN and flat fading, whereas the semi-analytical is derived for the frequency-selective case. As depicted in the figure, the SINRs of all subcarriers in OFDM are identical. On the contrary, the SINR of P-OFDM varies as a function of the subcarrier index. Such results imply that CFO may cause severe BER degradation for P-OFDM because the overall system BER is dominated by subcarriers with low SINRs. The figure also shows that SINR is different for OFDM and P-OFDM in AWGN and flat fading, yet it is the same for frequency-selective fading. Moreover, the obtained simulation results match the analytical/semi-analytical (A/S) results very well, where the semi-analytical results correspond to the
frequency-selective channel obtained using (24). It is worth noting that $d_0$ and $d_1$ are immune to CFO in AWGN channel where $\text{SINR}_{d_0} = \text{SINR}_{d_1} = \text{SNR}$.

Fig. 2 shows $\text{SINR}_k$ for selected subcarriers versus $\varepsilon$ for P-OFDM in AWGN, flat, and frequency-selective fading channels, and it also shows $\text{SINR}$ for OFDM. As can be noted from the figure, not only the SINR degradation depends on the subcarrier index in the P-OFDM, but also the sensitivity to CFO. For example, $d_{16}$ has the worst SINR and it is the most sensitive to CFO among the considered subcarriers, regardless of the channel conditions. The SINR for any subcarrier is determined by $\beta_{k,k}$ given in (21), where increasing $\beta_{k,k}$ increases the signal power and reduces the interference, and vice versa. For $d_{16}$, $\beta_{16,16}$ has the lowest value for the frequency-selective channel case, and thus, it has the largest interference and lowest SINR. Moreover, it is worth noting that increasing the number of subcarriers $N$ does not affect the SINR for any subcarrier in AWGN and flat fading channels due to the block nature of the matrix $M$, which maintains $\beta_{k,k}$ and the number of interfering terms for the $k$th subcarrier fixed regardless the value of $N$. In frequency-selective channels, it can be demonstrated numerically using (24) that increasing $N$ may improve $\text{SINR}_k$ for certain values of $k$, and decrease it for other values.

As can be noted from the Fig. 2, $\text{SINR} = \text{SNR}$ when $\varepsilon = 0$ for the AWGN channel, whereas it is not the case for the flat and frequency-selective channels, which is due to the equalization process. More specifically, $\text{SINR} \leq \text{SNR}$ for flat and frequency-selective channels when $\varepsilon = 0$. As confirmed by the analysis, the SINR of both OFDM and P-OFDM does not depend on the equalization process in AWGN channels. The SINR difference between P-OFDM and OFDM, and the difference between $\text{SINR}_k$ for each subcarrier is due to the structure of the WHT transform, which is represented by $\Psi^1_{k,m,p,v,i}$ and $\Psi^2_{k,m,p,n,v,i}$ in (24). As $\text{sign} \{ \Psi^1, \Psi^2 \} \in \pm 1$, the summations in the numerator and denominator of (24) may increase or decrease based on the specific value of $k$. Interestingly, the interference terms in the AWGN and flat fading channels cancel each other for $k = 0, 1$. In OFDM, $\text{sign} \{ \Psi^1, \Psi^2 \} = 1$, the summations of the numerator and denominator of (24) are independent of $k$, and thus, the OFDM is a special case of the P-OFDM as described in Appendix I.

The average BER versus SNR of P-OFDM for selected subcarriers and the average BER for OFDM and P-OFDM systems for the three considered channels with $\varepsilon = 0.1$ are shown in Figs. 3 and 4. The BER results shown in the figures confirm the results in Fig. 1, where the P-OFDM subcarriers that have SINR higher than the conventional OFDM have a better BER performance. It is noteworthy that the average BER of P-OFDM in frequency-selective channels is better than the conventional OFDM due to the frequency diversity gain provided by the precoding process. Since such gain does not exist in AWGN and flat fading channels, the BER of the conventional OFDM is better than P-OFDM, which is more sensitive to CFO.

Fig. 5 shows the average BER versus SNR of both conventional OFDM and P-OFDM over AWGN, flat, and frequency-selective channels for different values of $\varepsilon$ where $\text{SNR} = 20 \text{ dB}$. It can be noticed from the figure that the BER of both systems degrades severely for high values of $\varepsilon$ regardless of the channel model, particularly for $\varepsilon \geq 0.05$. However, the relative BER performance between the conventional and P-OFDM depends significantly on the channel model. For example, the BER results over AWGN and flat fading channels presented in Fig. 5(a) and (b) show that conventional OFDM consistently outperforms P-OFDM. Such behavior can be justified by noting that P-OFDM...
Fig. 3. Average BER of conventional OFDM and P-OFDM systems in AWGN and Rayleigh flat fading channels, $\varepsilon = 0.1$.

The $\text{SINR}$ versus the normalized CFO for conventional OFDM with MMSE is shown in Fig. 6. As can be noted from the figure, the equalization process reduces $\text{SINR}$ even for $\varepsilon = 0$, which can be justified by noting that

$$\text{SINR}_{k|\varepsilon=0} = \frac{\mathbb{E}[\lambda_1]}{\mathbb{E}[\lambda_2]}$$

where $\mathbb{E}[\lambda_1] < \mathbb{E}[\lambda_2]$ given that $\mathbb{E}[|H_{k}|^2] = 1$ and $\text{SNR} \gtrsim 1.46 \text{ dB}$. For example, in Rayleigh fading channels with $\text{SNR} = 20 \text{ dB}$, $\text{SINR}_{k|\varepsilon=0} = 10 \log_{10} (\text{SNR}) - 5.25 = 14.73 \text{ dB}$. Consequently, the $\text{SINR}$ degradation becomes less sensitive to $\varepsilon$. To clarify this point, consider the case where $\varepsilon = 0.05$, which gives $\text{SINR}$ degradation of about $13.2\%$ without equalization, while it is $8.7\%$ for the equalized system. Therefore, analyzing the $\text{SINR}$ without equalization might be misleading because it amplifies the impact of the CFO on OFDM.

Figs. 7 and 8 present the simulated $\text{SINR}_{k}$ and BER using the Haar transform and ZCMT. The semi-analytical results using (24) matches the simulation results very well, but omitted to avoid figure congestion. As can be noted from Fig. 7, $\text{SINR}_{k}$ for both transforms is not uniform for all subcarriers, however, the differences for the Haar transform case are very small. Moreover, the distribution of $\text{SINR}_{k}$ for the ZCMT is different from the WHT. Although the WHT and ZCMT $\text{SINR}_{k}$ distribution is different, the BER results in Fig. 8 show that their BER sensitivity to CFO is equivalent. As expected, the BER of the Haar transform at $\varepsilon = 0$ is higher than the WHT and ZCMT, however, it is more immune at high values of $\varepsilon$, nevertheless, its BER advantage is insignificant.

V. C ONCLUSION

This paper presented the SINR analysis of P-OFDM in the presence of CFO and evaluated its impact on the BER. The considered P-OFDM system is based on the WHT, the conventional OFDM was used as a benchmark. The obtained analysis for the WHT was then generalized for the Haar and ZCMT. The average SINR for P-OFDM is analytically evaluated for the considered wireless channel models. Semi-analytical average SINR expression is obtained for the frequency-selective fading channel models, whereas exact expressions are presented for the flat Rayleigh fading and AWGN channels. The validity of our analysis was shown through the perfect match of the analytical and the simulated average SINR. The simulation results showed that the average SINR of the P-OFDM is significantly different in the presence of CFO, where the average SINR per subcarrier is dependent on the subcarrier index. The variable SINR per individual subcarrier in the P-OFDM system can be mitigated by considering adaptive bit loading to reduce the number of bits for subcarriers with low SINR given that the CFO value is known at the transmitter. The obtained results also showed that each of the considered precoding transforms have different sensitivities to CFO.
APPENDIX I
SINR ANALYSIS OF CONVENTIONAL OFDM

In a frequency-selective fading channel, the instantaneous SINR of the $m$th subcarrier can be defined as

$$\text{SINR}_m = \frac{\left| \hat{H}_m H_{m,m} a_m \right|^2}{\sum_{p=0, p \neq m}^{N-1} \left| \hat{H}_m H_{m,p} a_p \right|^2 + |\eta_m|^2}. \quad (43)$$

For the AWGN channel case, the channel frequency response $H_m = H_p = 1 \forall m$. Thus, SINR$_m$ can be simplified to

$$\text{SINR}_m = \frac{|\alpha|^2 |a_m|^2}{\sum_{p=0, p \neq m}^{N-1} |\alpha_{m,p} a_p|^2 + |\eta_m|^2}. \quad (44)$$
Therefore, the average SINR can be expressed as

\[
\text{SINR}_{m} = \frac{|\alpha|^2 \mathbb{E}[|a_m|^2]}{\mathbb{E}[\sum_{p=0}^{N-1} |\alpha_{m,p}|^2] + \mathbb{E}[\eta_m]^2]}
\]

(45)

which is similar to the results obtained in [25, eq. (28)].

In flat fading channels, the average SINR can be written as

\[
\text{SINR}_{m} = \frac{|\alpha|^2 \mathbb{E}[|\lambda_1|]}{\mathbb{E}[\sum_{p=0}^{N-1} \sigma_{m,p}^2 |\alpha_{m,p}|^2] + \mathbb{E}[\lambda_2] \mathbb{E}[|\eta_m|^2]}
\]

(51)

which after some straightforward manipulations can be written as

\[
\text{SINR}_{m} = \frac{|\alpha|^2 \mathbb{E}[|\lambda_1|]}{\mathbb{E}[\sum_{p=0}^{N-1} |\alpha_{m,p}|^2] + \frac{1}{\text{SNR}} \mathbb{E}[\lambda_2]}
\]

(52)

where \( \mathbb{E}[\lambda_1] \) and \( \mathbb{E}[\lambda_2] \) are given in (35) and (37), respectively.

REFERENCES


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