Effective Rate of MISO Systems over Fisher-Snedecor $\mathcal{F}$ Fading Channels

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Abstract—This paper investigates the effective rate (ER) of multiple-input single-output (MISO) systems over both independent and identically distributed (i.i.d.) and independent and non-identically distributed (i.n.i.d.) Fisher-Snedecor $\mathcal{F}$ fading channels with delay constraints. Novel closed-form expressions are derived for the ER of both aforementioned scenarios. Furthermore, for the i.i.d. case asymptotic simple closed-form expressions for the ER in high- and low-signal-to-noise-ratios, are provided. Physical insights about the impact of channel and system parameters on the ER are also revealed. An interesting finding is that the ER increases with the mean channel power, while decreases with the delay factor. Finally, the correctness of the analytical results is validated through Monte-Carlo simulations.

Index Terms—Fisher-Snedecor $\mathcal{F}$ distribution, delay constraint, effective rate, MISO systems.

I. INTRODUCTION

The performance of wireless communication systems can be improved by adopting multiple antennas at the transmitter, called as multiple input single output (MISO) systems. In this context, Shannon capacity has been investigated as a fundamental performance metric in several previous works. Alternatively to Shannon capacity the effective rate (ER) (or effective capacity) was proposed in [1] as a quality-of-service (QoS) metric, capable of taking the system’s delay constraint into account. In [2], the authors studied the ER, when the channel is subject to $\kappa$-$\mu$ shadowed fading. Furthermore, the authors in [3] considered the ER in multi-source and multi-destination cooperative networks over Rayleigh fading channels, while [4] conducted an analysis of the ER of a MISO system over several well-known fading channels, i.e., Rician, Nakagami-$m$, and Generalized-$\kappa$. However, these channels either model the shadowing by using the lognormal distribution, which renders most statistics of interest analytically intractable, or require the evaluation of complex special functions.

More recently, a composite fading model named Fisher-Snedecor $\mathcal{F}$ was proposed in [5], which shows accurate fit of experimental measurements for wireless body area networks (WBANs). Additionally, based on the experimental channel data provided therein, it was presented that the Fisher-Snedecor $\mathcal{F}$ fits better in most of the cases, compared to Generalized-$\kappa$ fading model. Importantly, its statistical characteristics are more tractable: the authors in [6] presented closed-form expressions for the sum of independent and non-identically distributed (i.n.i.d.) Fisher-Snedecor $\mathcal{F}$ random variables (RVs), while in [7], the physical layer security of the Wyner’s wiretap model in Fisher-Snedecor $\mathcal{F}$ fading was investigated. However, to the best of the authors’ knowledge, the ER of MISO systems over Fisher-Snedecor $\mathcal{F}$ fading channels is still not available in the open technical literature.

Motivated by the above considerations, this paper studies the ER of MISO systems over both i.i.d. and i.n.i.d. Fisher-Snedecor $\mathcal{F}$ fading channels. Novel closed-form formulas are derived, while in order to provide meaningful insights about the impact of the system and channel parameters on the ER, asymptotic simple and insightful closed-form expressions are derived. Based on these results, some useful guidelines for the design of MISO systems are provided.

II. SYSTEM MODEL

In this letter, a flat-fading MISO system is considered, where the transmitter is equipped with $n_T$ antennas. The received signal at the receiver is given by

$$y = h x + n, \quad (1)$$

where $h$ is a channel fading vector and $x$ is the transmit vector. It is assumed that the transmit power $P$ is equally allocated to each antenna, i.e., $\mathbb{E} \{ x x^\dagger \} = (P/n_T) I$, where $\mathbb{E} \{ \cdot \}$ means expectation and $I$ is the unit matrix.

As proposed in [8], the MISO system can be considered as a queuing system over block fading channels, where the packets arrive according to a stationary service process. It is further assumed that the transmitter sends uncorrelated complex Gaussian signals with zero-mean and only the receiver has available channel state information [3], [4]. The ER is
defined as [8, Eq. (3)]

\[
R(\rho, \theta) = -\frac{1}{A} \log_2 \left( \mathbb{E} \left\{ \left( 1 + \frac{\rho}{nT} hh^\dagger \right)^{-A} \right\} \right) \text{ bits/s/Hz.}
\]  

(2)

where \( \rho \) denotes the average signal-to-noise-ratio (SNR). Moreover, \( A \triangleq \theta TB/\ln 2 \) with \( \theta \) being the decay factor, \( T \) is the block length and \( B \) is the system bandwidth.

Fisher-Snedecor \( \mathcal{F} \) distribution characterizes the simultaneous occurrence of multipath fading and shadowing, in which the mean square (RMS) power of a received Nakagami-\( m \) signal is assumed to be subjected to variations induced by an inverse Nakagami-\( m \) RV. The probability density function (pdf) is given by \[5, Eq. (5)]

\[
f_\mathcal{F}(\omega) = \frac{m^m(m\Omega)^{m/2} \omega^{m-1}}{B(m, m)(m\omega + m\Omega)^{m+m/2}}, \tag{3}
\]

where \( \Omega \triangleq \mathbb{E} \{ \omega \} \) is the mean power, \( m \) and \( m_s \) denote the fading severity and shadowing parameters, and \( B(\cdot, \cdot) \) is the beta function \[9, Eq. (8.384.1)].

III. I.I.D. FISHER-SNEDECOR \( \mathcal{F} \) FADING CHANNELS

A. Effective Rate

We assume that the elements of the channel vector, \( h \), follow i.i.d. Fisher-Snedecor \( \mathcal{F} \) distribution. From the definition, it can be readily observed that the sum of \( M \) i.i.d. Fisher-Snedecor \( \mathcal{F} \) RVs follows also a Fisher-Snedecor \( \mathcal{F} \) distribution, with parameters \( Mm \), \( Mm_s \) and \( M\Omega \). Thus, by using (3) along with some algebraic manipulations, the pdf of

\[
\mathcal{H} = \sum_{\ell=1}^{nT} |h_\ell|^2\]

can be written as

\[
f_{\text{i.i.d.}}(\omega) = \frac{m^{nm}(nm\Omega)^{n/2} \omega^{n-1}}{B(nm, nm)(nm\omega + nm\Omega)^{nm+n/2}}. \tag{4}
\]

Substituting (4) into (2), and by using \[9, Eq. (3.197.1)], a closed-form expression for the ER over MISO i.i.d. Fisher-Snedecor \( \mathcal{F} \) fading channels is

\[
R_{\text{i.i.d.}}(\rho, \theta) = -\frac{1}{A} \log_2 \left[ \frac{B(nm, m + nm_s)}{B(nm, nm_s)} \right] \times 2F_1 \left( A, nm; A + nm (m + m_s); 1 - \frac{nm_s\Omega}{m} \right), \tag{5}
\]

where \( 2F_1 (\cdot, \cdot; \cdot; \cdot) \) denotes the Gauss hypergeometric function \[9, Eq. (9.100)]. Note that this is a build-in function in most of the standard software packages (e.g., Matlab and Mathematica).

Although (5) is in closed-form, it includes special functions and thus, offers little insights into the physical impacts of the system and channel parameters on the ER. Therefore, it would be useful to investigate the ER in both high- and low-SNR regimes.

B. Asymptotic Effective Rate

In high-SNRs with \( \rho \to \infty \), (2) can be written as

\[
R(\rho, \theta) \approx -\frac{1}{A} \log_2 \left( \mathbb{E} \left\{ \left( \frac{\rho}{N_t} hh^\dagger \right)^{-A} \right\} \right). \tag{6}
\]

Substituting (4) into (6) and using \[9, Eq. (3.194.3)], a simple closed-form expression for the ER at high-SNR can be derived as

\[
R_{\text{h-SNR}}(\rho, \theta) = -\frac{1}{A} \log_2 \left( \frac{B(nm, m - A, nm_s + A)}{B(nm, nm_s)} \right) + \log_2 \left( \frac{m_s\Omega}{m} \right) + \log_2 \rho. \tag{7}
\]

The diversity gain denotes the increase in the slope of \( R_{\text{h-SNR}} \) against the average SNR, \( \rho \), which is evidently equal to 1 in this case. Also, it is worthwhile pointing out that the ER achieved as \( \rho \to \infty \) is a monotonically increasing function of the number of antennas, \( nT \), and the shadowing parameter, \( m_s \), due to the diversity contribution from the multi-antennas. Additionally, the shadowing of the RMS signal power in the Fisher-Snedecor \( \mathcal{F} \) fading channels does not exist when \( m_s \to \infty \), and (7) reduces to Nakagami-\( m \) fading.

The ER of Fisher-Snedecor \( \mathcal{F} \) fading channels in the low-SNR regime, can be written in terms of the normalized transmit energy per information bit, i.e., \( E_b/N_0 \), as \[10, Eq. (9)]

\[
R(\frac{E_b}{N_0}) \approx S_0\log_2 \left( \frac{E_b/N_0}{E_b/N_0_{\text{min}}} \right), \tag{8}
\]

where \( S_0 \) denotes the rate against the SNR slope, and \( E_b/N_0_{\text{min}} \) is the minimum \( E_b/N_0 \) required for reliably communication at any nonzero rate. As mentioned in \[10\], \( S_0 \) and \( E_b/N_0_{\text{min}} \) can be derived from the first and second derivatives of \( R(\rho, \theta) \) as \( \rho \to 0 \) through

\[
\frac{E_b}{N_0_{\text{min}}} \Delta = \frac{1}{R'(0, \theta)}, \tag{9}
\]

\[
S_0 \Delta = -2 \ln \left( \frac{R''(0, \theta)}{R'(0, \theta)} \right)^2, \tag{10}
\]

where \( R'(0, \theta) \) and \( R''(0, \theta) \) are given in \[8, Eqs. (16-17)].

Since \( \mathbb{E} \{ |h_\ell|^2 \} = \Omega \), it is easy to obtain that \( \mathbb{E} \{ hh^\dagger \} = nt\Omega \). With the help of \[5, Eq. (9)]\], the fourth moment of \( |h_\ell|^2 \) is given by

\[
\mathbb{E} \left\{ |h_\ell|^4 \right\} = \left( \frac{m\Omega}{m} \right)^2 \frac{B(m + 2, m_s - 2)}{B(m, m_s)}. \tag{11}
\]

Therefore,

\[
\mathbb{E} \left\{ (hh^\dagger)^2 \right\} = \sum_{i=1}^{nT} \mathbb{E} \left\{ |h_i|^4 \right\} + \sum_{i=1}^{nT} \sum_{j=1, j \neq i}^{nT} \mathbb{E} \left\{ |h_i|^2 |h_j|^2 \right\} = nt \left( \frac{m\Omega}{m} \right)^2 \frac{B(m + 2, m_s - 2)}{B(m, m_s)} + nt (nT - 1) \Omega^2. \tag{12}
\]

With the aid of (9) and (10), holds that

\[
\frac{E_b}{N_0_{\text{min}}} = \frac{\ln 2}{\Omega}, \tag{13}
\]

\[
S_0 = -2 \left[ \frac{A - (A + 1)}{nt} \times \left( \frac{m\Omega}{m} \right)^2 \frac{B(m + 2, m_s - 2)}{B(m, m_s)} + nt - 1 \right]^{-1}. \tag{14}
\]

Similar to the high-SNR case, the low-SNR slope \( S_0 \) increases with both \( nT \) and \( m_s \), while it monotonically decreases.
as $A$ increases, that is, the delay factor $\theta$, block length $T$, or system bandwidth $B$ increases. Also note that $E_b/N_{0,\min}$ is a decreasing function in the mean power $\Omega$ and is independent of other channel parameters (e.g., the fading severity parameter $m$ and the shadowing parameter $m_s$) or the delay constraint $\theta$.

IV. I.N.I.D. FISHER-SNEDECOR $\mathcal{F}$ FADING CHANNELS

Let us consider a MISO system over i.n.i.d. Fisher-Snedecor $\mathcal{F}$ fading channels. The pdf of the sum of the i.n.i.d. Fisher-Snedecor $\mathcal{F}$ RVs is given by \cite[Eq. (5)]{6}

$$f_{\text{i.n.i.d.}}(\omega) = \frac{\omega^{N_t-1} \Gamma \left( \sum_{\ell=1}^{N_t} m_{\ell} \right)}{\Gamma \left( \sum_{\ell=1}^{N_t} m_{\ell} \right)} \prod_{\ell=1}^{N_t} \left( \frac{m_{\ell}}{m_{s,\ell} \Omega_{\ell}} \right) \frac{m_{\ell} \Gamma (m_{\ell} + m_{s,\ell})}{\Gamma (m_{s,\ell})}$$

$$\times f_B \left( m_1 + m_{s,1}, \ldots, m_{n_T} + m_{s,n_T}, m_1, \ldots, m_{n_T}; \sum_{\ell=1}^{n_T} m_{\ell}; -m_1/m_{s,1} \Omega_1, \ldots, -m_{n_T}/m_{s,n_T} \Omega_{n_T} \right), \quad \omega \geq 0,$$

(15)

where $f_B(\cdot)$ represents the Lauricella multivariate hypergeometric function \cite[Eq. (1.10.2)]{11}.

Lemma. A closed-form expression for the ER in the i.n.i.d. case can be derived as in (16) at the bottom of this page, where $H^{0,1;1,\ldots,1;1,\ldots,1}_{0;2,1,\ldots,2;1}$ denotes the multivariate Fox $H$-function \cite[Def. (A.1)]{12}.

Proof: Please see Appendix.

Note that, by rewriting the multivariate Fox $H$-function in (16) in the form of Mellin-Barnes integrals and using \cite[Eqs. (8.384.1, 9.113)]{9} and \cite[Defs. (A.20, A.31)]{12}, the ER for the i.n.i.d. case can reduce to that for the i.i.d. case (5).

V. NUMERICAL RESULTS AND DISCUSSION

In order to validate the theoretical analysis, we perform Monte-Carlo simulations by generating $10^5$ independent Fisher-Snedecor $\mathcal{F}$ channel realizations. The ER against delay factor $\rho$ for the block length $T$ is depicted in Fig. 1. A perfect agreement can be observed between the analytical results (5) and those obtained by Monte-Carlo simulations. As anticipated, the ER monotonically decreases as delay factor $\theta$ and block length $T$ increases as well because the initial decay-rate of the MISO system is proportional to block length $T$ \cite{13}. Additionally, the gap between the curves is narrowing, when the corresponding block length $T$ increases, which implies that the longer $T$ is, the more pronounced its effect will be.

![Fig. 1. Analytical and simulated ER against delay factor $\theta$ with different block length $T$.](image1)

Fig. 1: High- and low-$E_b/N_0$ approximated and simulated ER against $E_b/N_0$ with different combination of $n_T$ and $\Omega$ for i.i.d. MISO Fisher-Snedecor $\mathcal{F}$ fading ($T=1$ s, $B=1$ MHz, $\theta=1$ 1/bits, $\Omega=1$, $m=2$, and $m_s=30$).

![Fig. 2. High- and low-$E_b/N_0$ approximated and simulated ER against $E_b/N_0$ with different combination of $n_T$ and $\Omega$.](image2)

Fig. 2: High- and low-$E_b/N_0$ approximated and simulated ER against $E_b/N_0$ with different combination of $n_T$ and $\Omega$ for i.i.d. MISO Fisher-Snedecor $\mathcal{F}$ fading ($T=1$ s, $B=1$ MHz, $\theta=1$ 1/bits, $\Omega=1$, $m=2$, and $m_s=30$).

![Fig. 3.](image3)

Fig. 3 illustrates the ER, along with its high-$E_b/N_0$ approximation (from (7)) by using the expression $E_b/N_0 = \rho/(\pi T R(\rho, \theta))$, and its low-$E_b/N_0$ approximation, as in (8). It is readily concluded that better ER performance can be achieved by increasing the number of antennas $n_T$. Also, an increase in the average power $\Omega$ may get the ER large but
leaves the required $E_b/N_0_{\text{min}}$ value unaffected. Moreover, the approximation of the ER is considerably tight for all the cases. It is worth mentioning that in the high-$E_b/N_0$ case, the accuracy is improved by larger values of $n_T$ and $S_0$, which is on the contrary with case in the low-$E_b/N_0$.

In Fig. 3, we plot the analytical and simulated ER of MISO i.n.i.d. Fisher-Snedecor $F$ fading channels against average SNR $\rho$. For different combination of $n_T$ and $A$ for the case of i.n.i.d. MISO Fisher-Snedecor $F$ fading ($\Omega = 1$, $m = \{2, 1.5, 1.75, 2.5\}$), and $m_s = \{25, 30.5, 27.5, 35.75\}$.

VI. CONCLUSIONS

In this paper, the impact of delay on the ER of MISO systems over Fisher-Snedecor $F$ fading channels is investigated. When no delay constraint is imposed, i.e., the delay approaches to infinity, the ER tends to the Shannon capacity. Novel closed-form expressions for the ER over both i.i.d. and i.n.i.d. $F$ fading channels are derived. To obtain physical insights, a simple closed-form expression for the ER at high-SNR regime was also presented. Additionally, tractable expressions are derived at low-SNR regime. The results are beneficial for the design of practical MISO systems.

APPENDIX

By employing [11, Eq. (1.10.10)], (15) is written in the form of multiple Barnes-type contour integrals as [6, Eq. (21)]

$$f_{\text{i.n.i.d.}}(\omega) = \prod_{\ell = 1}^{n_T} \left( \frac{m_\ell}{m_{s_\ell} \Omega_\ell} \right) \omega \left( \sum_{\ell = 1}^{n_T} m_\ell \right)^{-1} \frac{1}{(2\pi)^{n_T}} \int \cdots$$

$$\int_{C_{n_T}} \Gamma \left( \sum_{\ell = 1}^{n_T} (m_\ell + s_\ell) \right) \Gamma \left( \frac{m_\ell}{m_{s_\ell} \Omega_\ell} \omega \right) ds_1 ds_2 \cdots ds_{n_T}. \tag{17}$$

Substituting (17) into (2), and with the aid of [9, Eq. (3.194.3)], we can derive (18). The proof is completed by expressing (18) in terms of the multivariate Fox $H$-function as (16), which can be efficiently implemented in Matlab [14].

REFERENCES