New Analytical Framework for the Products of Independent RVs With Wireless Applications

Yunfei Chen
School of Engineering
University of Warwick
Coventry, United Kingdom CV4 7AL
Email: Yunfei.Chen@warwick.ac.uk

George K. Karagiannidis
Dept. of Electrical and Computer Engineering
Aristotle University of Thessaloniki
GR-54124, Greece
Email: geokarag@auth.gr

Hao Lu, Ning Cao
Computer and Information
Hohai University
Nanjing, China, 210098
Email: caoning@vip.163.com

***Abstract*— A novel analytical framework for evaluating the statistics of the products of independent random variables is proposed. Compared with other methods which use either an infinite series or a special function, the new method provides simple and efficient closed-form approximations in terms of elementary functions, such as powers and exponentials, and therefore, is very easy to implement. The accuracy of the new approximation is examined. Numerical results show that it is quite accurate in most regions of interest. As an application, these new approximations are used in wireless communications theory to derive novel closed-form expressions for the outage probability of cascaded fading channels. Numerical examples show that the newly derived closed-form expressions provide insights on the behavior of important performance metrics as the outage probability, the bit error rate and the channel capacity.**

I. INTRODUCTION

Usually, in the analysis of systems, the statistics of the product of independent random variables (RVs) is required. For example, in a multi-hop wireless relaying system using amplify-and-forward protocol with fixed amplification factor for statistically independent hops, the cascaded channel from source to destination can be modeled as the product of the RVs that describe the channel gains for the individual hops [1]. Moreover, in a multiple input multiuple output (MIMO) keyhole system, the electromagnetic wave propagates through several keyholes such that the overall channel gain can also be modeled as the product of the RVs that describe the individual keyhole channels [2]. The product of RVs also arises in several cases, where the performances of wireless communications systems are considered. For example, the outage probability of the received signal in a multi-hop wireless relaying system is a function of the product of the channel gains for all the hops [3].

Previous works on the statistics of the product of independent RVs include the following: In [4] and [5], the probability density function (PDF) of the product of several independent Gaussian RVs was derived in the form of an infinite series and the special Meijer-G fuction, respecitvely. In [6], the PDF of the product of independent RVs was derived for the H-function distribution. In [7], the PDF of the product of independent Rayleigh RVs was derived in terms of both the special Meijer-G function and the infinite series. In [3], the PDF of the product of independent Nakagami-m RVs was derived using the special Meijer-G function, while in [8], the PDF of the product of independent generalized Nakagami-m RVs was also derived using the special Meijer-G function. Note that the infinite series based method gives a simple expression of the PDF. However, in order to calculate the PDF, the infinite series has to be truncated, and the number of truncated terms has to be determined heuristically for each number of independent RVs in the product, making it difficult to implement in software. On the other hand, the special function method gives a closed-form expression of the PDF without truncation. However, the special functions as the Meijer-G and the H-functions are essentially contour integrals and the solution of contour integral is time-consuming. In conclusion, both the infinite series method and the special function method are computationally complicated.

In this paper, an efficient analytical framework is proposed as a simpler method to approximately evaluate the PDF of the product of several independent RVs. Unlike the infinite series method, the new approximate PDF is derived in closed-form so that no truncation is required. Also, unlike the special function method, the new approximate PDF is expressed in terms of elementary functions, such as powers and the exponentials, so that no contour integration or table look-up are needed. The generalized Gamma and Gaussian distributions are studied, while numerical results show that the newly derived approximation works very well in most regions of interest. As an application, these new approximate PDFs are also employed to calculate the outage probability, the bit error rate and the channel capacity of the multi-hop wireless relaying system.

II. SYSTEM MODEL

Consider the product

$$Y'_s = X'_{1,s} X'_{2,s} \cdots X'_{n,s} \tag{1}$$

where $X'_{1,s}, X'_{2,s}, \cdots, X'_{n,s}$ are n independent RVs and s indicates the specific name of the distribution.

For the generalized Gamma distribution, one has the PDF of $X'_{i,GM}$ as [9]

$$f_{X'_{i,GM}}(x) = \frac{vm^m}{\Gamma(m)\Omega_i^{mv}} x^{mv-1} e^{-\frac{mx^v}{\Omega_i^v}}, x > 0 \tag{2}$$

where $i = 1, 2, \cdots, n$, $\Gamma(\cdot)$ is the complete Gamma function, GM denotes the generalized Gamma distribution, $m > 0.5$ is the fading figure, $\Omega_i > 0$ is the scale parameter and $v > 0$ is the shape parameter. Note that [9] named (2) as the $\alpha-\mu$ distribution while we name it as the generalized Gamma distribution to be consistent with the original work by Stacy. The two distributions are equivalent. By normalizing the RV with respect to $\Omega_i/m^{1/v}$ as $X_{i,GM} = X'_{i,GM}/(\Omega_i/m^{1/v})$, one has

$$f_{X_{i,GM}}(x) = \frac{v}{\Gamma(m)} x^{mv-1} e^{-x^v}, x > 0. \tag{3}$$

When $m = 1$, (3) corresponds to the Weibull distribution.

For the generalized Gaussian distribution, one has the PDF of $X'_{i,GA}$ as [10]

$$f_{X'_{i,GA}}(x) = \frac{p\sqrt{\Gamma(3/p)}}{2\sigma_i\Gamma(1/p)\sqrt{\Gamma(1/p)}} e^{-\frac{|x|^p}{\sigma_i^p}\left(\frac{\Gamma(3/p)}{\Gamma(1/p)}\right)^{p/2}} \tag{4}$$

where $+\infty > x > -\infty$, GA denotes the generalized Gaussian distribution, p is the shape parameter, σ_i^2 is the average power and the location parameter has been omitted for simplicity. If one normalizes X'_i with respect to $\sigma_i/\sqrt{\Gamma(3/p)/\Gamma(1/p)}/2^{1/p}$ as $X_{i,GA} = X'_{i,GA} * 2^{1/p}\sqrt{\Gamma(3/p)/\Gamma(1/p)}/\sigma_i$, one has

$$f_{X_{i,GA}}(x) = \frac{p}{2^{1+1/p}\Gamma(1/p)} e^{-\frac{1}{2}|x|^p}, +\infty > x > -\infty. \tag{5}$$

For later use, the product of the independent normalized RVs is defined as

$$Y_s = X_{1,s}X_{2,s}\cdots X_{n,s}. \tag{6}$$

The normalization factor is taken such that the argument in the exponential function has a scaling factor of 1. In the above, it is assumed that different RVs have different scale parameters or average powers while they have the same shape parameters.

III. New Approximations

Starting from the approximation proposed in [11] for the n-th order root of the product of independent Rayleigh RVs, it has been shown in [11] that for

$$Z'_{RA} = (X'_{1,RA}X'_{2,RA}\cdots X'_{n,RA})^{\frac{1}{n}} \tag{7}$$

where RA denotes the Rayleigh distribution and $X'_{i,RA}$ is the i-th Rayleigh RV with distribution

$$f_{X'_{i,RA}}(x) = xe^{-\frac{x^2}{2}}, x > 0 \tag{8}$$

the PDF of Z'_{RA} can be well approximated as a Nakagami-m distribution with

$$f_{Z'_{RA}}(x) \approx 2\left(\frac{m_0}{\Omega_0}\right)^{m_0} \frac{1}{\Gamma(m_0)} x^{2m_0-1} e^{-\frac{m_0}{\Omega_0}x^2}, x > 0 \tag{9}$$

where $m_0 = 0.6102 * n + 0.4263$ and $\Omega_0 = 0.8808 * n^{-0.9661} + 1.12$ are determined heuristically using the MATLAB distribution fitting tool from 10^6 simulated values of Z'. Although this method is empirical and does not represent rigorous mathematical derivation, numerical results will show that it provides accurate approximation and therefore, offers engineering solution. Using (9), the PDF of the product of independent normalized exponential RVs can be approximated as

$$f_{Y_{EX}}(x) \approx \left(\frac{2m_0}{\Omega_0}\right)^{m_0} \frac{1}{n\Gamma(m_0)} x^{\frac{m_0}{n}-1} e^{-\frac{2m_0}{\Omega_0}x^{\frac{1}{n}}}, x > 0. \tag{10}$$

where EX denotes the exponential distribution. On the other hand, the exact PDF of the product of independent normalized exponential RV can be derived using a method similar to that in [4] and [5]. The Mellin transform of a function $f(x)$ is defined as

$$M\{s\} = E\{x^{s-1}\} = \int_0^\infty x^{s-1} f(x)dx \tag{11}$$

where $E\{\cdot\}$ represents the expectation operation and $x > 0$. Consider the normalized exponential RV. Its Mellin transform is given by

$$M_{X_{i,EX}}(s) = \Gamma(s). \tag{12}$$

A very useful characteristic of this transform is that the Mellin transform of the product of independent RVs equals to the product of the Mellin transforms of the individual RVs. Therefore, the Mellin transform of the product of n independent normalized exponential RVs can be derived from (12) as

$$M_{Y_{EX}}(s) = \Gamma^n(s). \tag{13}$$

From (13), the exact PDF of the product of n independent normalized exponential RVs is derived by taking an inverse Mellin transform as

$$f_{Y_{EX}}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma^n(s)ds \tag{14}$$

where the path of integration is any line parallel to the imaginary axis lying within the strip of analyticity of $\Gamma^n(s)$. In the infinite series method, the contour integral in (14) can be solved by using the calculus of residues. In the special function method, the contour integral in (14) can be rewritten in terms of the Meijer-G function or the H-function. In this paper, an approximation is proposed by using (10) and (14) as

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma^n(s)ds \approx \left(\frac{2m_0}{\Omega_0}\right)^{m_0} \frac{1}{n\Gamma(m_0)} x^{\frac{m_0}{n}-1} e^{-\frac{2m_0}{\Omega_0}x^{\frac{1}{n}}}, x > 0. \tag{15}$$

A. Generalized Gamma

Consider the generalized Gamma distribution first. From (11), the Mellin transform of a PDF actually equals the $(s-1)$-th order moment of the RV. Using [12, eq. (3)], one has the Mellin transform of the generalized Gamma PDF as

$$M_{X_{i,GM}}(s) = \frac{\Gamma(m + \frac{s-1}{v})}{\Gamma(m)}. \tag{16}$$

Thus, the Mellin transform of the product of n independent normalized generalized Gamma RVs can be derived from (16) as

$$M_{Y_{GM}}(s) = \frac{\Gamma^n(m + \frac{s-1}{v})}{\Gamma^n(m)}. \tag{17}$$

Then, the exact PDF of the product of n independent normalized generalized Gamma RVs can be calculated as the inverse Mellin transform

$$f_{Y_{GM}}(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}\frac{\Gamma^n(m+\frac{s-1}{v})}{\Gamma^n(m)}ds. \quad (18)$$

Let $s'=m+(s-1)/v$. One can simplify (18) as

$$f_{Y_{GM}}(x)=\frac{vx^{mv-1}}{\Gamma^n(m)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(x^v)^{-s'}\Gamma^n(s')ds'. \quad (19)$$

Finally, by using (15) in (19), taking an integration of the resulting function over x from 0 to ∞, and normalizing the function with respect to the integral, one has the approximation to the PDF of the product of n independent normalized generalized Gamma RVs as

$$f_{Y_{GM}}(x) \approx \frac{v}{n\Gamma(m_0+mn-n)}\left(\frac{2m_0}{\Omega_0}\right)^{m_0+mn-n} x^{\frac{vm_0}{n}+mv-v-1}e^{-\frac{2m_0}{\Omega_0}x^{\frac{v}{n}}}, x>0. \quad (20)$$

In the above, the normalization with respect to the integral of the resulting function from 0 to ∞ takes the approximation error into account so that the overall area under the PDF equals to 1. The approximation to the cumulative distribution function (CDF) of the product of n independent normalized generalized Gamma RVs can be derived by integrating (20) over x from 0 to y as

$$F_{Y_{GM}}(y)\approx\gamma\left(m_0+mn-n,\frac{2m_0}{\Omega_0}y^{\frac{v}{n}}\right) \quad (21)$$

where

$$\gamma(a,x)=\frac{1}{\Gamma(a)}\int_0^x e^{-t}t^{a-1}dt \quad (22)$$

is the lower incomplete Gamma function. Note that (22) is different from what is defined in [13, eq. (8.350.1)] because of the extra normalization over $\Gamma(a)$. Using (20) and (21), one can derive the PDF and CDF for the product of n independent unnormalized generalized Gamma RVs by performing simple variable transformation.

B. Generalized Gaussian

In the case of the generalized Gaussian distribution, the method is slightly different from the above because the generalized Gaussian RV takes both positive and negative values. In this case, as shown in [14] and [4], this difficulty can be overcome by splitting the positive and the negative components of the generalized Gaussian PDF to define two functions as

$$f_{X_{i,GA}^+}(x)=\frac{p}{2^{1+1/p}\Gamma(1/p)}e^{-\frac{1}{2}|x|^p}S(x) \quad (23)$$

and

$$f_{X_{i,GA}^-}(x)=\frac{p}{2^{1+1/p}\Gamma(1/p)}e^{-\frac{1}{2}|x|^p}S(-x) \quad (24)$$

where $S(x)$ is the step function with $S(x)=1$ for $x>0$ and $S(x)=0$ otherwise. From [15], the Mellin transform of $f_{X_{i,GA}^+}(x)$ is given by

$$M_{X_{i,GA}^+}(s)=\frac{2^{\frac{s-1}{p}-1}}{\Gamma(1/p)}\Gamma(s/p). \quad (25)$$

Then, using [4, eq. (18)], the Mellin transforms of the positive component and the negative component of the product of n independent normalized generalized Gaussian RVs are given by

$$M_{Y_{GA}^+}(s)=M_{Y_{GA}^-}(s)=2^{n-1}\frac{2^{n\frac{s-1}{p}-n}}{\Gamma^n(1/p)}\Gamma^n(s/p) \quad (26)$$

where the coefficient of 2^{n-1} takes into account the fact that $f_{X_{i,GA}^+}(x)$ only describes half of the possible values for the generalized Gaussian RV.

From (26), the exact PDF of the positive component of the product of independent normalized generalized Gaussian RVs is derived as

$$f_{Y_{GA}^+}(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}2^{n-1}\frac{2^{n\frac{s-1}{p}-n}}{\Gamma^n(1/p)}\Gamma^n(s/p)ds. \quad (27)$$

By taking a variable transformation of $s'=s/p$, (27) can be simplified as

$$f_{Y_{GA}^+}(x)=\frac{2^{-n/p-1}p}{\Gamma^n(1/p)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{y^p}{2^n}\right)^{-s'}\Gamma^n(s')ds'. \quad (28)$$

Using (15) in (28), taking an integration of the resulting function over x from 0 to ∞ and normalizing the resulting function with respect to the integral multiplied by 0.5, as $f_{Y_{GA}^+}(x)$ only defines the PDF in half of the plane, one has the approximate PDF of the positive component of the product of independent normalized generalized Gaussian RVs as

$$f_{Y_{GA}^+}(x) \approx \frac{p}{2n\Gamma(m_0-n+n/p)}\left(\frac{m_0}{\Omega_0}\right)^{m_0-n+n/p} x^{pm_0/n-p}e^{-\frac{m_0}{\Omega_0}x^{p/n}}, x>0. \quad (29)$$

Combining with the negative component, the approximate PDF of the product of independent normalized generalized Gaussian RVs is

$$f_{Y_{GA}}(x) \approx \frac{p}{2n\Gamma(m_0-n+n/p)}\left(\frac{m_0}{\Omega_0}\right)^{m_0-n+n/p} |x|^{pm_0/n-p}e^{-\frac{m_0}{\Omega_0}|x|^{p/n}}, +\infty>x>-\infty. \quad (30)$$

By integrating $f_{Y_{GA}}(x)$ over x from $-\infty$ to y, one has the CDF of the product of independent normalized generalized Gaussian RVs as

$$F_{Y_{GA}}(y)\approx 0.5+0.5\gamma\left(m_0-n+n/p,\frac{m_0}{\Omega_0}y^{\frac{p}{n}}\right), y>0 \quad (31)$$

and

$$F_{Y_{GA}}(y)\approx 0.5-0.5\gamma\left(m_0-n+n/p,\frac{m_0}{\Omega_0}(-y)^{\frac{p}{n}}\right), y<0. \quad (32)$$

IV. Application

In this section, the approximate PDFs and CDFs derived in the previous section are adopted to calculate the outage probability for multi-hop wireless relaying with cascaded fading channels. Similar analyses for other channel models have been conducted in [16] and [17]. For multi-hop wireless relaying systems with cascaded fading channels, the end-to-end signal-to-noise ratio (SNR) is given by [3, eq. (7)]

$$\alpha = \frac{E_s}{N_0}\prod_{i=1}^{n}|X'_{i,s}|^2 \tag{33}$$

where E_s/N_0 is the transmitted SNR and $X'_{i,s}$ represents the fading coefficient of the i-th hop. Define

$$\bar{\alpha} = \frac{E_s}{N_0}\prod_{i=1}^{n}E\{|X'_{i,s}|^2\} \tag{34}$$

as the average end-to-end SNR. By normalizing (33) with respect to the average end-to-end SNR in (34), one has

$$\tilde{\alpha} = \prod_{i=1}^{n}\frac{|X'_{i,s}|^2}{E\{|X'_{i,s}|^2\}}. \tag{35}$$

The outage probability is defined as the probability that $\tilde{\alpha}$ is below a certain threshold α_{th}, that is,

$$P_o = Pr\{\tilde{\alpha} < \alpha_{th}\}. \tag{36}$$

Using the approximate PDF or CDF derived in the previous section and after some mathematical manipulations, one has

$$P_o = \gamma\left(m_0 + mn - n, \frac{2m_0}{\Omega_0}\alpha_{th}^{\frac{v}{2n}}\frac{\Gamma^{\frac{v}{2}}(m+2/v)}{\Gamma^{\frac{v}{2}}(m)}\right) \tag{37}$$

for the generalized Gamma and

$$P_o = \gamma\left(n/p, \frac{2m_0}{\Omega_0}\alpha_{th}^{\frac{p}{2n}}\frac{\Gamma^{\frac{p}{2}}(3/p)}{\Gamma^{\frac{p}{2}}(1/p)}\right) \tag{38}$$

for the generalized Gaussian. One sees that the new approximate PDF or CDF of the product of the generalized RVs allow us to have a closed-form expression for the outage probability. The incomplete Gamma function only involves with a one-dimensional real integral with finite limits, unlike the Meijer-G function and the H-function that involve with a complex contour integral with infinite limits. Therefore, it is relatively simple to calculate or tabulate.

V. Numerical Results and Discussion

In this section, numerical results are presented. In these examples, the approximate result is calculated using our new approximations while the exact result is calculated using the contour integrals or the Meijer-G function method. Our tests show that the new approximations are valid for a wide range of parameters, although due to the limited space, we are only able to show some sample comparisons.

Figs. 1 and 2 compare the new approximate CDFs for the products of independent generalized Gamma and generalized Gaussian RVs, respectively, with the exact CDFs from the Meijer G function method. One sees that the approximate CDFs match quite well with the exact CDFs in most cases considered. For the generalized Gamma RV, the approximate CDFs and the exact CDFs are graphically indistinguishable for all values of y examined when $m = v = 1$. When $m = v = 2$, the difference between the approximate CDF and the exact CDF is noticeable but it decreases with the value of y. For the generalized Gaussian RV, the difference between the approximate CDF and the exact CDF is negligible when the value of x is large. On the other hand, when the value of x is small and $p = 2$, there is considerable disagreement between the approximate PDF and the exact PDF. Therefore, the approximate CDF of the product of independent generalized Gaussian RVs is most useful when the value of p is small and the argument considered is large.

Figs. 3 and 4 show the outage probability performance of the wireless relaying system in cascaded generalized Gamma and generalized Gaussian fading channels, respectively. As expected, the outage probability increases when the SNR threshold increases. In the extreme case when the SNR thresh-

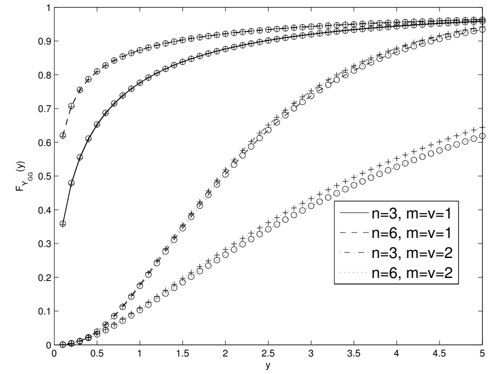

Fig. 1. Comparison of the approximate CDF (plus line) and the exact CDF (circle line) for different distribution parameters and different numbers of independent generalized Gamma RVs.

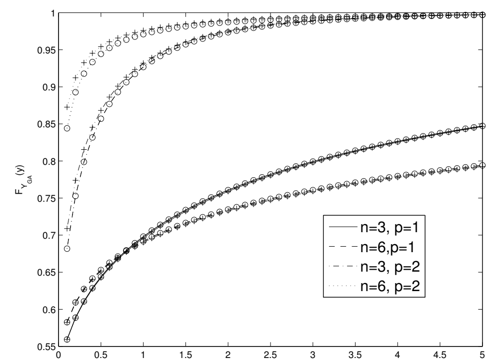

Fig. 2. Comparison of the approximate CDF (plus line) and the exact CDF (circle line) for different distribution parameters and different numbers of independent generalized Gaussian RVs.

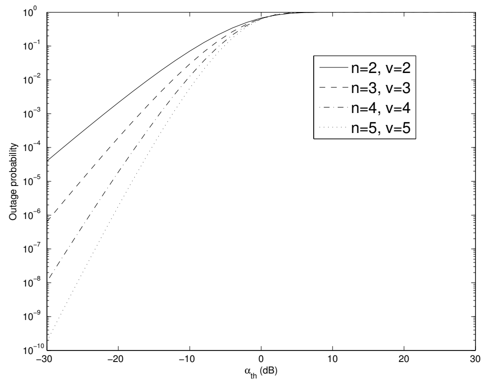

Fig. 3. The outage probability for different distribution parameters and different numbers of independent RVs in cascaded generalized Gamma fading channels when $m = 2$.

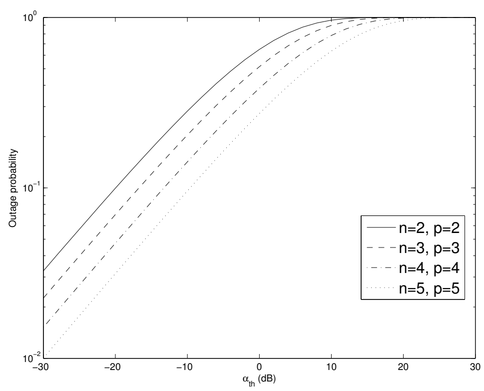

Fig. 4. The outage probability for different distribution parameters and different numbers of independent RVs in cascaded generalized Gaussian fading channels.

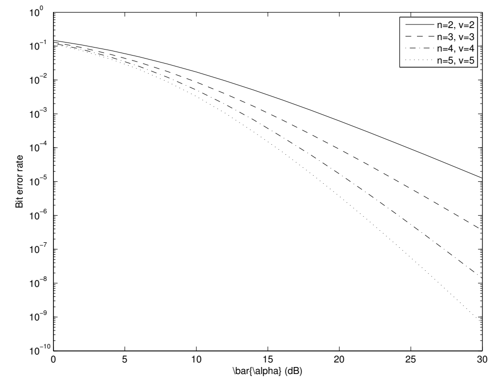

Fig. 5. The bit error rate for different distribution parameters and different numbers of independent RVs in cascaded generalized Gamma fading channels when $m = 2$.

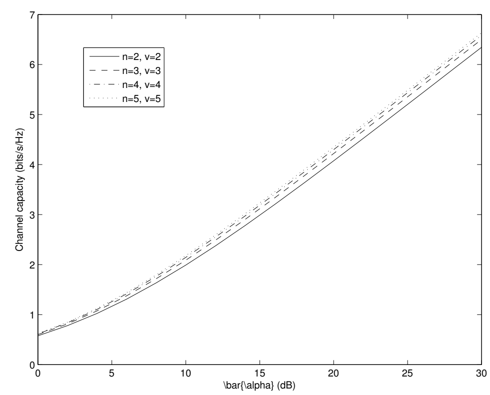

Fig. 6. The capacity for different distribution parameters and different numbers of independent RVs in cascaded generalized Gamma fading channels when $m = 2$.

old goes to infinity, the outage probability will approach 1. The outage probabilities for the generalized Gamma increases much faster than the outage probability for the generalized Gaussian, when the SNR threshold increases. This can be explained from (37) and (38). Recall that the lower incomplete Gamma function is defined in (22). By taking differentiations of P_o with respect to α_{th}, one can easily find that it is the parameter of a in the lower incomplete Gamma function that determines the slope of the curve when $p = v$. Since $a = n/p$ for the generalized Gaussian channel is smaller than $a = m_0 + mn - n$ for the generalized Gamma in the cases considered, one has a smaller slope for the generalized Gaussian channel.

Figs. 5 and 6 use the generalized Gamma distribution as an example to show the average bit error rate and the average channel capacity in a cascaded channel. One sees that the average bit error rate decreases when the value of $n = v$ increases, while the average channel capacity increases when the value of $n = v$ increases. The difference of the bit error rate is small when the signal-to-noise ratio is small and is large when the signal-to-noise ratio is large, while the difference of the channel capacity is small for all values of signal-to-noise ratio considered.

VI. Conclusions

New approximate PDFs and CDFs for the products of independent generalized Gamma and generalized Gaussian RVs have been derived. Calculations based on these new approximations are much simpler than those using the existing methods, although they are limited to independent RVs with the same shape parameters. Numerical results have shown that the new approximations also have satisfactory accuracies. Using these new approximations, new closed-form expressions for the outage probability in the wireless cascaded channel have been obtained and examined. The outage probability behavior has also been explained using the derived expressions based on the new approximate PDFs and CDFs, which is otherwise not possible using expressions based on the existing methods.

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