Gallager's Error Exponent Analysis of STBC Systems over η-μ Fading Channels

Jiayi Zhang*†, Michail Matthaiou‡, George K. Karagiannidis§, Zhenhui Tan*†, and Haibo Wang*

*Institute of Broadband Wireless Mobile Communications, Beijing Jiaotong University, Beijing, China
†State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University, Beijing, China
‡Department of Signals and Systems, Chalmers University of Technology, Gothenburg, Sweden
§Department of Electrical and Computer Engineering, Aristotle University of Thessaloniki, Thessaloniki, Greece
Email: jayizhang@bjtu.edu.cn, michail.matthaiou@chalmers.se, geokarag@auth.gr, {zhhtan, hbwang}@bjtu.edu.cn

***Abstract*—The Gallager's random coding error exponent for space-time block codes (STBC) over multiple-input multiple-output (MIMO) block-fading channels, with Gaussian input distribution, is investigated. Gallager's error exponent can be used to determine the required codeword length to achieve a prescribed error probability at a given rate below the channel capacity. We first provide new, analytical expressions for Gallager's exponent of STBC systems over η-μ fading channels. The Shannon capacity and cutoff rate, which can be directly derived from Gallager's exponent, are further examined. In order to get additional insights, a high signal-to-noise ratio analysis is pursued to investigate the effects of coherence time and codeword length on the error probability. For the sake of completeness, we provide the link to previous known results on Rayleigh and Nakagami-m fading channels.**

I. Introduction

As one of the most important technologies, MIMO is a bandwidth-efficient scheme for next generation wireless communications [1]–[3]. Although ergodic Shannon capacity has been used as the typical metric to evaluate the performance of MIMO systems (see e.g., [4], [5] and references therein), it gives only the maximum achievable information rate and may not be sufficient to reflect the fundamental limits of MIMO communication systems. Naturally, the error probability tends to zero as the block length tends to infinity, when the rate is less than the channel capacity [6]. However, the block length cannot be long enough because of the limitations on delay and encoding/decoding complexity. For this reason, an information-theoretic metric, namely Gallager's random coding exponent (or reliability function), has been proposed in [7] to exploit this fundamental tradeoff. As an easily computable lower bound, this metric can determine the probability of error, P_e, as a function of the codeword length, L, and the information rate, R.

Since then, several works have investigated Gallager's exponent of various single-antenna communication systems over flat fading channels [8], [9]. Moreover, [10] derived Gallager's exponent for MIMO systems over Rayleigh block-fading channels with spatial fading correlation and subject to an average power constraint, assuming perfect channel state information (CSI) at the receiver and no CSI at the transmitter. The relationship between Gallager's exponent, information rate, codeword length and signal-to-noise ratio (SNR) for fast Rayleigh fading MIMO-ARQ channels was examined in [11].

Yet, the results for Gallager's exponent of multiple-antenna systems with STBC in non-Rayleigh fading channels are still few. Only recently, Xue *et al.* [12] investigated Gallager's exponent of STBC systems over Nakagami-m fading channels. However, the Nakagami-m fading model relies on the assumption of a homogeneous scattering environment, which is often unrealistic since surfaces are spatially correlated in most propagation environments [13]. Motivated by this limitation, the η-μ distribution was proposed in [14] to model non-homogeneous environments. In [14], it was shown that this fading model can provide better fit to experimental data than the Rayleigh and Nakagami-m fading models, while it involves both models as special cases. Motivated by these important observations, we herein analytically investigate Gallager's exponent of STBC systems over η-μ fading channels.[1]

The remainder of the paper is organized as: Following a brief overview of the MIMO system model and Gallager's exponent used throughout the paper in Section II, analytical expressions for Gallager's exponent of STBC over MIMO η-μ fading channels along with a detailed high-SNR analysis are presented in Section III. Moreover, a set of numerical results is presented to explore the implications of the model parameters on the proposed theoretical expressions. Finally, our significant findings are summarized in Section IV.

Notation: We use upper and lower case boldface letters to denote matrices and vectors, respectively. The symbol $(\cdot)^\dagger$ represents the Hermitian transpose, the trace operator of a square matrix is denoted by $\mathrm{tr}(\cdot)$, $\mathrm{etr}(\cdot) = e^{\mathrm{tr}}(\cdot)$, while $\|\cdot\|_F$ denotes the matrix Frobenius norm. The expectation operator of a random variable is given by $\mathtt{E}\{\cdot\}$, the matrix determinant reads as $\det(\cdot)$, while $\lceil\cdot\rceil$ denotes the ceiling operation to the nearest integer.

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[1]Note that an extended journal version of this paper can be found in [15].

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II. System model and Gallager's Exponent

A. System model

We consider a single-user MIMO system with N_t transmit antennas and N_r receive antennas, whose complex input-output relationship can be expressed as

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{N} \quad (1)$$

where $\mathbf{H} \in \mathbb{C}^{N_r \times N_t}$ denotes the fading channel matrix with entries $h_i\,(i = 1, 2, \ldots, N_t N_r)$, while $\mathbf{X} \in \mathbb{C}^{N_t \times N_c}$ is the transmit matrix containing N_c symbols. Also, $\mathbf{Y} \in \mathbb{C}^{N_r \times N_c}$ represents the received signal matrix and $\mathbf{N} \in \mathbb{C}^{N_r \times N_c}$ is the complex zero-mean additive white Gaussian noise (AWGN) matrix with the variance of elements being N_0. Considering N_b independent coherence intervals, the block codeword length of a reliable communication link is $N_b N_c$. Moreover, the input signal matrix is subject to an average power constraint of the form $\mathtt{E}\left\{\mathrm{tr}\left(\mathbf{X}\mathbf{X}^\dagger\right)\right\} = N_c \mathrm{tr}\left(\mathbf{Q}\right) \leq N_c P$, where $\mathbf{Q}$ is the $N_t \times N_t$ positive semidefinite input covariance matrix and P is the total transmit power. Assuming that the transmitter has no CSI, it is meaningful to assume that uniform power allocation is being performed across the transmit antennas, such that $\mathbf{Q} = \frac{P}{N_t}\mathbf{I}$.

For STBC, the MIMO channel can be represented as $N_t \times N_r$ parallel single-input-single-output channels for each data symbol [16]. Thus, the effective output symbol SNR is given by [5, Eq. (4)]

$$\gamma_o = \frac{\gamma}{N_t}\left\|\mathbf{H}\right\|_F^2 \quad (2)$$

where $\gamma = \frac{P}{R_c N_0}$ is the transmit SNR and R_c is the information code rate. For the sake of clarity and without loss of generality, we assume full-rate STBC such that $R_c = 1$.

B. Gallager's exponent

1) Random coding exponent: With maximum-likelihood decoding, an upper bound on the error probability of MIMO channels with continuous inputs and outputs can be obtained as [7]

$$P_e \leq \left(\frac{2e^{r\delta}}{\xi}\right)^2 \exp\left(-N_b N_c E_r\left(p_{\mathbf{X}}\left(\mathbf{X}\right), R, N_c\right)\right) \quad (3)$$

where $p_{\mathbf{X}}\left(\mathbf{X}\right)$ is the input probability density function (p.d.f.). The above bound is given in terms of several arbitrary parameters, namely $r \geq 0$, $\delta \geq 0$, which are defined as

$$\xi \approx \frac{\delta}{\sqrt{2\pi N_b \sigma_\xi^2}} \quad (4)$$

$$\sigma_\xi^2 = \int_{\mathbf{X}} \left[\mathrm{tr}\left(\mathbf{X}\mathbf{X}^\dagger\right) - N_c P\right]^2 p_{\mathbf{X}}\left(\mathbf{X}\right) d\mathbf{X} \quad (5)$$

where ξ is estimated precisely for large N_b using the central limit theorem, while $\int_{\mathbf{X}}(\cdot)d\mathbf{X}$ denotes $\int_{\mathbf{X}_1}\int_{\mathbf{X}_2}\cdots\int_{\mathbf{X}_{N_b}}(\cdot)\,d\mathbf{X}_1 d\mathbf{X}_2 d\mathbf{X}_{N_b}$, with $\mathbf{X}_i$ being the transmitted sequence in the interval i. To achieve a desired error probability at an information rate R, the codeword length can be obtained by solving for N_b in (3) with $L = N_c \times \lceil N_b \rceil$.

In [7], the random coding exponent is defined as

$$E_r\left(p_{\mathbf{X}}\left(\mathbf{X}\right), R, N_c\right) = \max_{0\leq\rho\leq 1}\left(\max_{r\geq 0} E_0\left(p_{\mathbf{X}}\left(\mathbf{X}\right), \rho, r, N_c\right) - \rho R\right) \quad (6)$$

where

$$E_0\left(p_{\mathbf{X}}\left(\mathbf{X}\right), \rho, r, N_c\right) \triangleq -\frac{1}{N_c}\ln\left(\int_{\mathbf{H}} p_{\mathbf{H}}\left(\mathrm{H}\right)\int_{\mathbf{Y}}\left(p_{\mathbf{X}}\left(\mathbf{X}\right) \times e^{r\left[\mathrm{tr}\left(\mathbf{X}\mathbf{X}^\dagger\right) - N_c P\right]} p(\mathbf{Y}|\,\mathbf{X}, \mathbf{H})^{\frac{1}{1+\rho}} d\mathbf{X}\right)^{1+\rho} d\mathbf{Y} d\mathbf{H}\right). \quad (7)$$

In general, it is very difficult to minimize the upper bound by optimizing $p_{\mathbf{X}}\left(\mathbf{X}\right)$. Subject to the considered power constraint, we choose the capacity-achieving Gaussian distribution for $p_{\mathbf{X}}\left(\mathbf{X}\right)$ as [7]

$$p_{\mathbf{X}}\left(\mathbf{X}\right) = \pi^{-N_t N_c}\det\left(\mathbf{Q}\right)^{-N_c}\mathrm{etr}\left(-\mathbf{Q}^{-1}\mathbf{X}\mathbf{X}^\dagger\right). \quad (8)$$

This choice of the input distribution may facilitate the solution of (7), though it is optimal for the random coding exponent calculation only when the rate R approaches the channel capacity. Then, (7) becomes

$$\tilde{E}_0\left(\rho, \beta, N_c\right) \triangleq E_0\left(\frac{P}{N_t}\mathbf{I}_{\mathbf{N_t}}, \rho, r, N_c\right)\Bigg|_{\beta = N_t - rP} = \underbrace{\left(1+\rho\right)\left(N_t - \beta\right) + N_t\left(1+\rho\right)\ln\left(\beta/N_t\right)}_{A(\rho,\beta)} - \frac{1}{N_c}\ln\left(\mathtt{E}\left\{\left(1 + \frac{\gamma z}{\beta\left(1+\rho\right)}\right)^{-N_c\rho}\right\}\right) \quad (9)$$

where $z \triangleq \sum_{i=1}^{N_t N_r}\left|h_i\right|^2$. Then, the random coding exponent can be written as

$$E_r\left(R, N_c\right) = \max_{0\leq\rho\leq 1}\left(\max_{0\leq\beta\leq N_t}\tilde{E}_0\left(\rho, \beta, N_c\right) - \rho R\right). \quad (10)$$

As in [17], the error probability is given by

$$P_e \leq \frac{8\pi}{N_t}\left(N_t - \beta^*\left(\rho\right)\right)^2 N_b N_c e^{\left(2 - N_b N_c E_r\left(R, N_c\right)\right)} \quad (11)$$

where $\beta^*\left(\rho\right)$ is the value of β that maximizes $\tilde{E}_0\left(\rho, \beta, N_c\right)$, defined in (9) for each ρ, and is in the range $0 < \beta \leq N_t$.

2) Shannon capacity: From [18], the information rate R can be expressed as

$$R = \frac{\partial\tilde{E}_0\left(\rho, \beta^*\left(\rho\right), N_c\right)}{\partial\rho}. \quad (12)$$

Note that R becomes identical to the Shannon capacity $\langle C\rangle$ for $\rho = 0$ and $\beta^*\left(0\right) = N_t$, such that

$$\langle C\rangle = \frac{\partial\tilde{E}_0\left(\rho, \beta^*\left(\rho\right), N_c\right)}{\partial\rho}\Bigg|_{\rho=0, \beta^*(0)=N_t}. \quad (13)$$

3) Cutoff rate: As an important information-theoretic metric, the cutoff rate R_0 determines a lower bound to the Shannon capacity and the corresponding value of the zero-rate random coding exponent. By setting $\rho = 1$ and $\beta^*\left(1\right) = N_t$

in (9), the cutoff rate becomes [17]

$$R_0=\tilde{E}_0\left(1,\beta^*\left(1\right),N_c\right)=-\frac{1}{N_c}\ln\left(\mathtt{E}\left\{\left(1+\frac{\gamma z}{2N_t}\right)^{-N_c}\right\}\right). \tag{14}$$

III. Gallager's Exponent in η-μ Fading Channels

The η-μ distribution models the small-scale variation of the fading signal with the p.d.f. of the instantaneous SNR given in [14, Eq. (26)]

$$f_{\eta-\mu}\left(\omega\right)=\frac{2\sqrt{\pi}\mu^{\mu+\frac{1}{2}}h^{\mu}\omega^{\mu-\frac{1}{2}}}{\Gamma\left(\mu\right)H^{\mu-\frac{1}{2}}\Omega^{\mu+\frac{1}{2}}}e^{\left(-\frac{2\mu h\omega}{\Omega}\right)}I_{\mu-\frac{1}{2}}\left(\frac{2\mu H\omega}{\Omega}\right) \tag{15}$$

where $I_v\left(x\right)$ is the modified Bessel function of the first kind [19, Eq. (8.445)], $\Gamma\left(x\right)$ is the Gamma function [19, Eq. (8.310.1)], and $\Omega=\mathtt{E}\left\{\omega\right\}$ denotes the average power. The parameters h and H related to η are different in two formats. More specifically, according to format 1, $h=\left(2+\eta^{-1}+\eta\right)/4$ and $H=\left(\eta^{-1}-\eta\right)/4$, where $0<\eta<\infty$ is the scattered-wave power ratio between the in-phase and quadrature components. For format 2, $h=1/\left(1-\eta^2\right)$ and $H=\eta/\left(1-\eta^2\right)$, where $-1<\eta<1$ represents the correlation coefficient between the scattered-wave in-phase and quadrature components of each cluster of multipath. In both formats, the parameter μ denotes the half number of multipath clusters. Note that the Nakagami-m distribution can be obtained by setting $\mu=m$ and $\eta\to 0$ or $\eta\to\infty$ in the format 1 of the η-μ distribution. Moreover, the Rayleigh distribution is obtained by setting $\mu=0.5$ and $\eta=1$ in format 1 or $\mu=0.5$ and $\eta=0$ in format 2. Alternatively, it can be attained by setting $\mu=m/2$ and $\eta\to 1$ in format 1 or $\eta\to 0$ in format 2 [14]. Without loss of generality, we only consider format 1 in the following analysis.

According to [14], it is known that the sum of M independent and identically distributed squared η-μ RVs with parameters η, μ, and Ω is also an η-μ distribution with parameters η, $M\mu$ and $M\Omega$. As such, we can easily obtain the p.d.f. of z as follows,

$$p_{\eta-\mu}\left(z\right)=\frac{2\sqrt{\pi}h^{\mu N_tN_r}}{\Gamma\left(\mu N_tN_r\right)}\left(\frac{\mu}{\Omega}\right)^{\mu N_tN_r+\frac{1}{2}}\left(\frac{z}{H}\right)^{\mu N_tN_r-\frac{1}{2}}\times\exp\left(-\frac{2\mu hz}{\Omega}\right)I_{\mu N_tN_r-\frac{1}{2}}\left(\frac{2\mu Hz}{\Omega}\right). \tag{16}$$

By using the representation of $I_v\left(x\right)$ in terms of a generalized hypergeometric function ${}_pF_q\left(\cdot\right)$ [19, Eq. (9.238.2)], we can obtain an alternative expression for the p.d.f. of z as

$$p_{\eta-\mu}\left(z\right)=\frac{z^{2\mu N_tN_r-1}h^{\mu N_tN_r}}{\Gamma\left(2\mu N_tN_r\right)}\left(\frac{2\mu}{\Omega}\right)^{2\mu N_tN_r}e^{\left(-\frac{2\mu\left(h+H\right)}{\Omega}z\right)}\times{}_1F_1\left(\mu N_tN_r;2\mu N_tN_r;\frac{4\mu H}{\Omega}z\right). \tag{17}$$

A. Random coding exponent analysis

Based on the theoretical analysis presented in Section II, we first obtain the exact random coding exponent as follows:

Proposition 1: The random coding exponent of STBC over MIMO η-μ fading channels can be expressed as (18) at the bottom of this page, where $U(\cdot)$ is the Tricomi hypergeometric function [20, Eq. (13.1.3)].

Proof: We can directly substitute (16) into (9) and thereafter use the infinite series representation of $I_v\left(\cdot\right)$ from [19, Eq. (8.445.1)]. The involved integral is evaluated with the help of the following identity

$$\int_0^\infty\left(1+ax\right)^{-v}x^{q-1}e^{-px}dx=\frac{p^{v-q}}{a^v}\Gamma\left(q\right)U\left(v;v+1-q;\frac{p}{a}\right), \tag{19}$$

which is a combination of [21, Eq. (07.33.17.0007.01)] and [4, Eq. (39)]. Then, the proof concludes after invoking [20, Eq. (6.1.18)] and simplifications. ∎

In order to assess the convergence of the infinite series in (18), we assume that T_0-1 terms are used; hence, the associated truncation error Ξ_0 can be upper bounded as

$$\begin{aligned}\Xi_0&=\sum_{l=T_0}^{\infty}\frac{\Gamma\left(\mu N_tN_r+l\right)}{l!}\left(\frac{H}{h}\right)^{2l}\\&\times U\left(N_c\rho;N_c\rho-2\mu N_tN_r-2l+1;\frac{2\mu h\beta\left(1+\rho\right)}{\Omega\gamma}\right)\\&<U\left(N_c\rho;N_c\rho-2\mu N_tN_r-2T_0+1;\frac{2\mu h\beta\left(1+\rho\right)}{\Omega\gamma}\right)\\&\times\sum_{l=T_0}^{\infty}\frac{\Gamma\left(\mu N_tN_r+l\right)}{\Gamma\left(l+1\right)}\left(\frac{H}{h}\right)^{2l}\\&=U\left(N_c\rho;N_c\rho-2\mu N_tN_r-2T_0+1;\frac{2\mu h\beta\left(1+\rho\right)}{\Omega\gamma}\right)\\&\times\left(\frac{\Gamma\left(\mu N_tN_r\right)}{\left(1-\left(\frac{H}{h}\right)^2\right)^{-\mu N_tN_r}}-\sum_{l=0}^{T_0-1}\frac{\Gamma\left(\mu N_tN_r+l\right)}{\Gamma\left(l+1\right)}\left(\frac{H}{h}\right)^{2l}\right)\end{aligned} \tag{20}$$

$$E_r\left(R,N_c,\eta,\mu\right)=\max_{0\le\rho\le1}\max_{0\le\beta\le N_t}\left(A\left(\rho,\beta\right)-\frac{1}{N_c}\ln\left(\frac{h^{-\mu N_tN_r}}{\Gamma\left(\mu N_tN_r\right)}\left(\frac{2\mu h\beta\left(1+\rho\right)}{\Omega\gamma}\right)^{N_c\rho}\sum_{l=0}^{\infty}\frac{\Gamma\left(\mu N_tN_r+l\right)}{l!}\left(\frac{H}{h}\right)^{2l}\times U\left(N_c\rho;N_c\rho-2\mu N_tN_r-2l+1;\frac{2\mu h\beta\left(1+\rho\right)}{\Omega\gamma}\right)\right)\right)-\rho R. \tag{18}$$

where we have used the fact that $U(a; b-n; z)$ is a monotonically decreasing function in n, along with the definition of a hypergeometric function [19, Eq. (9.210.1)].

As was previously mentioned, our analysis elaborates also on the Shannon capacity and cutoff rate of STBC systems. We now present new results for these metrics over η-μ fading channels.

Corollary 1: The Shannon capacity of STBC over MIMO η-μ fading channels can be expressed as

$$\langle C\rangle = \frac{\exp\left(\frac{2\mu h N_t}{\Omega\gamma}\right)}{\Gamma(\mu N_t N_r)\, h^{\mu N_t N_r}} \sum_{l=0}^{\infty} \frac{\Gamma(\mu N_t N_r + l)}{l!} \left(\frac{H}{h}\right)^{2l} \times \sum_{n=1}^{2\mu N_t N_r + 2l} E_{2\mu N_t N_r + 2l + 1 - n}\left(\frac{2\mu h N_t}{\Omega\gamma}\right) \tag{21}$$

where $E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt$, $n = 0, 1, 2, \ldots$ and $\mathrm{Re}(x) > 0$ denotes the exponential integral of order n [20, Eq. (5.1.4)].

Proof: By plugging (16) into (13), we end up with the following integral expression

$$\langle C\rangle = \frac{2\sqrt{\pi} h^{\mu N_t N_r}}{\Gamma(\mu N_t N_r)} \sum_{l=0}^{\infty} \frac{H^{2l} (\mu/\Omega)^{2\mu N_t N_r + 2l}}{l!\,\Gamma(\mu N_t N_r + l + 1/2)} \times \int_0^\infty \ln\left(1 + \frac{\gamma z}{\beta(1+\rho)}\right) z^{2\mu N_t N_r + 2l - 1} e^{\left(-\frac{2\mu h z}{\Omega}\right)} dz. \tag{22}$$

With the aid of [4, Eq. (40)] and [10, Eq. (46)], we can evaluate the integral in (22) and arrive at the desired result in (21). ■

As for Proposition 1, we can prove the convergence of the infinite series in (22). By assuming $T_1 - 1$ terms are used, the associated truncation error Ξ_1 can be upper bounded as

$$\begin{aligned}\Xi_1 &= \sum_{l=T_1}^{\infty} \frac{\Gamma(\mu N_t N_r + l)}{l!} \left(\frac{H}{h}\right)^{2l} \times \sum_{n=1}^{2\mu N_t N_r + 2l} E_{2\mu N_t N_r + 2l + 1 - n}\left(\frac{2\mu h N_t}{\Omega\gamma}\right) \\ &< \nu E_\nu\left(\frac{2\mu h N_t}{\Omega\gamma}\right) \left(\Gamma(\mu N_t N_r)\left(1 - \left(\frac{H}{h}\right)^2\right)^{\mu N_t N_r} - \sum_{l=0}^{T_1 - 1} \frac{\Gamma(\mu N_t N_r + l)}{\Gamma(l+1)} \left(\frac{H}{h}\right)^{2l}\right)\end{aligned} \tag{23}$$

where $\nu = 2\mu N_t N_r + 2T_1 - 2$, and we have exploited the fact that $E_n(x)$ is a monotonically decreasing function in n.

Corollary 2: The cutoff rate R_0 of STBC over MIMO η-μ fading channels can be expressed as

$$R_0 = -\frac{1}{N_c} \ln \frac{h^{-\mu N_t N_r}}{\Gamma(\mu N_t N_r)} \left(\frac{4\mu h N_t}{\Omega\gamma}\right)^{N_c} \sum_{l=0}^{\infty} \frac{\Gamma(\mu N_t N_r + l)}{l!} \times \left(\frac{H}{h}\right)^{2l} U\left(N_c; N_c + 1 - 2\mu N_t N_r - 2l; \frac{4\mu h N_t}{\Omega\gamma}\right). \tag{24}$$

Proof: The proof follows a similar line of reasoning as in Proposition 1, by simply plugging (16) into (14) and using (19). ■

B. High-SNR analysis

Corollary 3: The random coding exponent of STBC over MIMO η-μ fading channels at high SNRs and for $N_c\rho < 2\mu N_t N_r$ can be expressed as (25) at the bottom of this page.

Proof: By considering the initial expression (9) and keeping only the dominant term therein as $\gamma \to \infty$, we can obtain the desired result in (25) with the aid of the following integral identity [19, Eq. (7.522.9)]

$$\int_0^\infty x^{\sigma-1} e^{-ax} {}_1F_1(\alpha; \beta; \lambda x)\, dx = \frac{\Gamma(\sigma)}{a^\sigma} {}_2F_1\left(\alpha, \sigma; \beta; \frac{\lambda}{a}\right) \tag{26}$$

where $\mathrm{Re}(\sigma) > 0$, $\mathrm{Re}(a) > \mathrm{Re}(\lambda)$. Note that the condition on the arguments of (26) is satisfied in our setting by taking $N_c\rho < 2\mu N_t N_r$. ■

Corollary 4: The Shannon capacity of STBC over MIMO η-μ fading channels at high SNRs can be expressed as

$$\langle C\rangle^\infty = \sum_{l=0}^{\infty} \frac{\Gamma(\mu N_t N_r + l)\, H^{2l}}{l!\,\Gamma(\mu N_t N_r)\, h^{\mu N_t N_r + 2l}} \times \left(\psi(2\mu N_t N_r + 2l) - \ln\left(\frac{2\mu h N_t}{\Omega\gamma}\right)\right) \tag{27}$$

where $\psi(\cdot)$ is Euler's digamma function [19, Eq. (8.360.1)].

Proof: The proof follows by taking γ large in (21), then using the integral identity [19, Eq. (4.352.1)] and simplifying the resulting expression. ■

Corollary 5: The cutoff rate of STBC over MIMO η-μ fading channels at high SNRs and for $N_c < 2\mu N_t N_r$ can be expressed as

$$R_0^\infty = -\ln\left(\frac{2\mu N_t\left(1 + \frac{1}{\eta}\right)}{\Omega\gamma}\right) - \frac{1}{N_c} \ln\left(\frac{\Gamma(2\mu N_t N_r - N_c)}{\Gamma(2\mu N_t N_r)}\right) - \frac{1}{N_c} \ln\left({}_2F_1\left(\mu N_t N_r, N_c; 2\mu N_t N_r; 1 - \frac{1}{\eta}\right)\right). \tag{28}$$

Proof: The proof concludes by following a similar line of reasoning as in Corollary 3. ■

$$E_r^\infty(R, N_c, \eta, \mu) = \max_{0\le\rho\le1} \max_{0\le\beta\le N_t} \left(A(\rho,\beta) - \rho \ln\left(\frac{\mu\beta\left(1+\eta^{-1}\right)(1+\rho)}{\Omega\gamma}\right) - \frac{1}{N_c} \ln\left(\frac{\Gamma(2\mu N_t N_r - N_c\rho)}{\Gamma(2\mu N_t N_r)} {}_2F_1\left(\mu N_t N_r, N_c\rho; 2\mu N_t N_r; 1 - \frac{1}{\eta}\right)\right)\right) - \rho R. \tag{25}$$

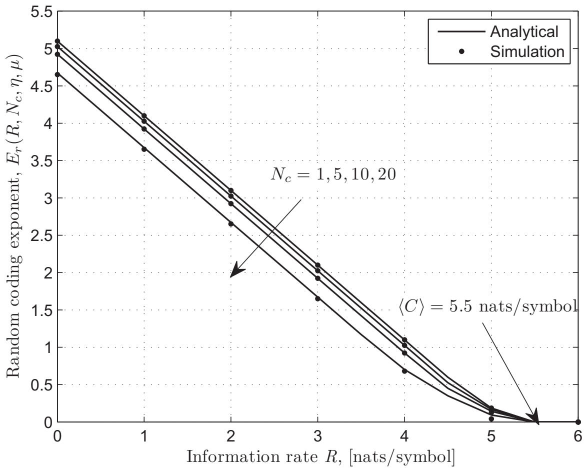

Fig. 1. Analytical and simulated random coding exponent against information rate for STBC systems over η-μ fading channels ($N_t = N_r = 4$, $\eta = 0.5$, $\mu = 1$, $\Omega = 2.5$, and $\gamma = 15$ dB).

C. Numerical results

In this subsection, the above theoretical analysis is validated through a set of Monte-Carlo simulations. We first generate 10^6 random realizations of the small-scale fading matrix $\mathbf{H}$ according to (16), and thereafter obtain the simulated random coding exponent, Shannon capacity, and cutoff rate via the corresponding expressions (18), (21), and (24), respectively.

In Fig. 1, the simulated random coding exponent is plotted along with the analytical expression (18). Note that the random coding exponent decreases monotonically with the parameter N_c, which means a less reliable communication can be achieved for a larger coherence time. Moreover, it is impossible to transmit any information at a positive rate with arbitrary small error probability when $N_c \to \infty$. As expected, the Shannon capacity is independent of N_c and represents the upper bound of R.

In order to get more insights into the effects of η and μ on coding requirements for STBC systems, the codeword length L required to achieve a fixed error probability and rate, i.e., $P_e \leq 10^{-6}$, $R = 4$ bits/symbol, is tabulated in Table I. We observe from Table I that for each value of γ, the codeword lengths for channels with small values of η and μ are much longer than those for channels with large values of η and μ. This is due to the advantages of having more multipath clusters. For example, the required codeword length for the case of $\eta = 0.2$ and $\mu = 0.2$ is almost 5.33 times the codeword length for the case of $\eta = 0.5$ and $\mu = 1$ for $\gamma = 12.5$ dB.

Figure 2 illustrates the simulated, analytical (24) and high-SNR approximation (28) cutoff rate as a function of the transmit SNR for STBC over MIMO η-μ fading channels. It shows that the cutoff rate is a monotonically increasing function of the transmit SNR, η and μ. Note that the effect of η on the cutoff rate is more pronounced than that of μ. We also note that as N_c increases, R_0 reduces to zero, while the Shannon capacity is independent of N_c. This difference reveals that the cutoff rate is more useful than the Shannon capacity in reflecting the reliability of block-fading channels, which is consistent with the results in [7] and [17].

TABLE I
REQUIRED CODEWORD LENGTHS OF STBC SYSTEMS OVER η-μ FADING CHANNELS AT A RATE $R = 4$ BITS/SYMBOL (2.77 NATS/SYMBOL) WITH $P_e \leq 10^{-6}$, $N_t = N_r = 2$, $\Omega = 2.5$, AND $N_c = 5$

SNR γ (dB)	$\mu = 0.2$ $\eta = 0.2$	$\mu = 0.5$ $\eta = 1$	$\mu = 1$ $\eta = 0.5$
10	460	110	70
12.5	160	45	30
15	85	30	20
17.5	55	20	15
20	35	15	10

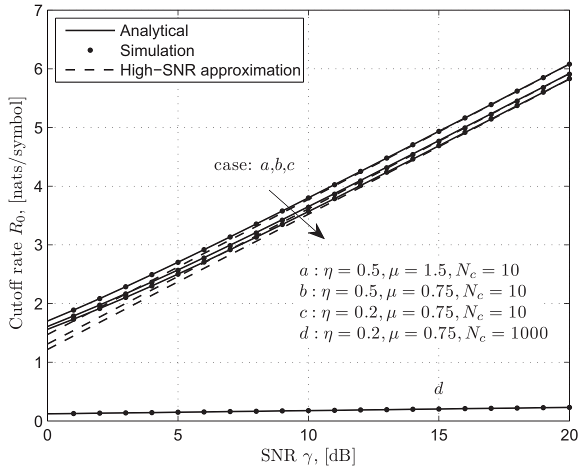

Fig. 2. Analytical, simulated and high-SNR approximation cutoff rate against the transmit SNR for STBC systems over η-μ fading channels ($N_t = N_r = 4$ and $\Omega = 2.5$).

D. Special cases

In this subsection, we provide simplified analytical expressions for some widely used fading models, namely Nakagami-m and Rayleigh. Note that all subsequent results are presented with no proof since the mathematical manipulations involved are straightforward. The link to previously reported results, where available, is also provided.

1) Nakagami-m fading channels: By setting $\mu = m/2$ and $\eta = 1$ in the η-μ distribution, we can obtain Nakagami-m distribution. Then, the Gallager's exponent in (18) reduces to

$$E_r(R, N_c, 1, m) = \max_{0 \leq \rho \leq 1} \max_{0 \leq \beta \leq N_t} \Bigg(A(\rho, \beta) \times -\frac{1}{N_c} \ln \left(\left(\frac{m\beta(1+\rho)}{\Omega\gamma} \right)^{N_c\rho} \times U\left(N_c\rho; N_c\rho + 1 - mN_tN_r; \frac{m\beta(1+\rho)}{\Omega\gamma} \right) \right) \Bigg) - \rho R. \quad (29)$$

Note that (29) coincides with [12, Eq. (23)] after applying the transformation [22, Eq. (8.4.46.1)].

We can also derive the Shannon capacity and cutoff rate expressions of STBC over Nakagami-m fading channels as:

$$\langle C\rangle = \exp\left(\frac{mN_t}{\Omega\gamma}\right)\sum_{n=1}^{mN_tN_r} E_{mN_tN_r+1-n}\left(\frac{mN_t}{\Omega\gamma}\right) \tag{30}$$

$$R_0 = -\frac{\ln\left(\left(\frac{2mN_t}{\Omega\gamma}\right)^{N_c} U\left(N_c; N_c+1-mN_tN_r; \frac{2mN_t}{\Omega\gamma}\right)\right)}{N_c}. \tag{31}$$

Note that (31) coincides with [12, Eq. (30)] after applying the transformation [22, Eq. (8.4.46.1)].

2) Rayleigh fading channels: When considering the Rayleigh distribution, we simply set $\mu = 1$ and $\eta = 1$ in the η-μ distribution. Therefore, the random coding exponent of η-μ fading channels in (18) reduces to

$$E_r(R, N_c) = \max_{0\le\rho\le 1}\max_{0\le\beta\le N_t}\left(A(\rho,\beta) - \frac{1}{N_c}\ln\left(\left(\frac{\beta(1+\rho)}{\Omega\gamma}\right)^{N_c\rho} \times U\left(N_c\rho; N_c\rho+1-N_tN_r; \frac{\beta(1+\rho)}{\Omega\gamma}\right)\right)\right) - \rho R.$$

Note that the Shannon capacity and cutoff rate expressions for STBC over Rayleigh fading channels can be written as

$$\langle C\rangle = \exp\left(\frac{N_t}{\Omega\gamma}\right)\sum_{n=1}^{N_tN_r} E_{N_tN_r+1-n}\left(\frac{N_t}{\Omega\gamma}\right) \tag{32}$$

$$R_0 = -\frac{1}{N_c}\ln\left(\left(\frac{2N_t}{\Omega\gamma}\right)^{N_c} U\left(N_c; N_c+1-N_tN_r; \frac{2N_t}{\Omega\gamma}\right)\right).$$

Note that (32) coincides with [23, Eq. (20)].

IV. Conclusion

For STBC systems over η-μ fading channels, a detailed Gallager's exponent analysis, which is the fundamental tradeoff between communication reliability and information rate, was presented. In particular, we derived new, exact analytical expressions for the random coding exponent, which extend and complement several previous results on Rayleigh and Nakagami-m fading channels. Moreover, the Shannon capacity and cutoff rate expressions were derived from Gallager's exponent. Furthermore, we drew significant insights into the reliability-rate tradeoff in MIMO systems, pertaining to the required codeword length to achieve a certain error probability. Finally, simplified high-SNR closed-form expressions of the above performance metrics were presented, which provided additional physical insights into the implications of several parameters (e.g., fading parameters, coherence time) on the required codeword lengths for a prescribed error probability.

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