Low-Complexity PHY-Layer Network Coding for Two-Way Compute-and-Forward Relaying

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***Abstract*—We present a novel low-complexity technique that obtains the Physical-Layer Network Coding (PNC) equation coefficient vectors for the two-way relay channel when Compute-and-Forward is employed. The proposed method is based on pre-computed look-up tables that are used for all channel realizations. It is shown that the size of the look-up tables can be made small by taking into account the statistics of the channel coefficients as well as power and performance specifications. Moreover, a low-complexity algorithm is developed for efficient real-time selection of the equation coefficient vectors using the instantaneous channel coefficients and the look-up tables. Although the method may at times exclude some candidate vectors from the search space, simulation results indicate that the effect on the achievable computation rate at the relay is very small. Hence, significant complexity reduction is achieved, while the computation rate remains extremely close to the optimal value.**

***Index Terms*—two-way relay channel, compute-and-forward, PHY-layer network coding**

I. Introduction

Network coding (NC) has been widely investigated as a promising candidate for the uplink of wireless networks. The concept behind NC is to combine different source messages at the relays into a common network-coded message, which is then forwarded to the desired destinations and is used to extract the original messages [1], [2]. When linear network coding (LNC) is employed, intermediate nodes send a linear combination of their received messages [3]. A special case of NC is Physical Layer (PHY-Layer) Network Coding (PNC), which was introduced in [4] and makes use of the additive nature of simultaneously arriving messages at the destination through the wireless channel. Compute-and-Forward (CoF), which was first introduced in [5], employs PNC at the relays and utilizes the algebraic structure of lattice codes [6] in order to decode a linear combination of the transmitted messages, using the observed channel coefficients. The relay chooses which integer linear combination of the messages will be decoded, with the aim to maximize the achievable rate at which it can decode this combination without errors, referred to as computation rate.

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In CoF networks, the choice of the equation coefficients – based on the maximization of the computation rate – has no analytical solution, but is an optimization problem, as described in [7]. Various works have proposed algorithmic solutions to this problem, using LLL lattice reduction [7], geometric programming [8] and a modified Fincke-Pohst method [9], [10], among others. However, although it is sometimes possible to reduce the number of candidate equation coefficient vectors that are included in the search (as in [9], [10]), all these methods are computationally demanding, since they should perform an on-line optimization of the candidate vector sets.

In this work, we propose a novel low-complexity PNC method for the two-way relay channel (TWRC) [11], [12], which is based on the use of look-up tables. The efficiency and novelty of the method is summarized in the following:

- The set of candidate equation coefficient vectors is drastically reduced since it is proven that the elements of the optimal vector are coprime.
- An algorithm that constructs the look-up tables offline is proposed, using the statistics of the channel coefficients and some desired power and accuracy specifications, which further reduce the candidate set. Thus, there is no need for complex online optimization of the PNC.
- A second algorithm that performs very low-complexity search over the look-up tables during the operation of the system is also proposed. For each channel realization, the algorithm selects the equation coefficient vector from the look-up tables, by performing simple comparisons using the channel state information (CSI).
- Simulation results indicate that in comparison with the exact solution of the optimization problem, the achievable computation rate is extremely close to the optimal value, while achieving very low complexity.

II. System Model

We consider a standard TWRC with two sources, equipped with one antenna each and denoted by S_1 and S_2 respectively,

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that exchange their messages through a CoF relay, denoted by R. According to the CoF scheme in [5], each source attempts to transmit an information vector $\mathbf{w}_l \in \mathbb{F}_p^k$, $l = 1, 2$, where k is the length of the vector and $\mathbb{F}_p$ is a finite field of p elements, with p a prime number. Each source encodes the length-k message $\mathbf{w}_l$ to a length-n lattice codeword $\mathbf{x}_l \in \mathbb{R}^n$, satisfying a power constraint $\|\mathbf{x}_l\|^2 \leq nP$.

Transmission is divided into two phases. In the first phase, both sources transmit their codewords towards the relay, which receives

$$\mathbf{y}_r = h_1\mathbf{x}_1 + h_2\mathbf{x}_2 + \mathbf{z}_r, \tag{1}$$

where $h_l \in \mathbb{R}^+$, $l = 1, 2$, are the channel coefficients between the source S_l and the relay R that are assumed constant during the transmission of a codeword, and $\mathbf{z}_r \in \mathbb{R}^n$ is the Additive White Gaussian Noise (AWGN) vector, which follows the normal distribution $\mathcal{N}(\mathbf{0}, \mathbf{I}_{n\times n})$, where $\mathbf{0}$ is the zero vector of length n and $\mathbf{I}_{n\times n}$ is the unitary matrix of size n. Thus, the Signal-to-Noise Ratio (SNR) is $SNR = P$. Although the method can be extended to the case where the channel coefficients also take negative values and follow other distributions, due to space limitations in this work, in the following we assume real non-negative channel coefficients that represent the envelope of a complex Gaussian channel, and are therefore Rayleigh distributed with probability density function (pdf) given by[1]

$$f_h(x) = \tfrac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, \quad x \geq 0. \tag{2}$$

The relay attempts to decode a combination of the transmitted messages, which is itself a message $\mathbf{w}_r \in \mathbb{F}_p^k$,

$$\mathbf{w}_r = q_1\mathbf{w}_1 \oplus q_2\mathbf{w}_2, \tag{3}$$

where $q_1, q_2 \in \mathbb{F}_p$ and $\oplus$ denotes addition over the field $\mathbb{F}_p$. This is achieved by decoding a lattice codeword

$$\mathbf{x}_r = [a_1\mathbf{x}_1 + a_2\mathbf{x}_2] \bmod \Lambda, \tag{4}$$

where $a_1, a_2 \in \mathbb{Z}$ with $q_l = [a_l] \bmod p$ and $\bmod \Lambda$ denotes the modulo operation over the coarse lattice Λ of the lattice code, so that $\mathbf{x}_r$ also belong to the same codebook used by the sources S_1 and S_2 [5], [6].

In the second transmission phase, the relay retransmits the lattice codeword $\mathbf{x}_r$ towards S_1 and S_2. The received signals are respectively

$$\mathbf{y}_l = \tilde{h}_l\mathbf{x}_r + \mathbf{z}_l, \quad l = 1, 2, \tag{5}$$

where $\tilde{h}_l$ is the channel coefficient from the relay to S_l and $\mathbf{z}_l \in \mathbb{R}^n$ are AWGN vectors with $\mathcal{N}(\mathbf{0}, \mathbf{I}_{n\times n})$. Note that the parameters $\tilde{h}_l$ can be the same with h_l or different, and are known to the respective S_l. The source S_l, $l = 1, 2$, decodes the message $\mathbf{w}_r$ and then its desired message, since it has knowledge of the coefficients q_l and of the message $\mathbf{w}_l$.

We denote the channel coefficient vector by $\mathbf{h} = [h_1, h_2]^T$ and the equation coefficient vector by $\mathbf{a} = [a_1, a_2]^T$, where $[\cdot]^T$ denotes transposition. The computation rate region achieved by the relay for a given $\mathbf{a}$ was proven to be [5]

$$\mathcal{R}_c(\mathbf{a}) = \frac{1}{2}\log_2^+\left[\left(\|\mathbf{a}\|^2 - \frac{P\left(\mathbf{h}^T\mathbf{a}\right)^2}{1 + P\|\mathbf{h}\|^2}\right)^{-1}\right], \tag{6}$$

where $\log_2^+(\cdot) = \max(0, \log_2(\cdot))$ and $\|\cdot\|$ denotes the Euclidean norm. Accordingly, the transmission rate region achieved by the source S_l, $l = 1, 2$, is

$$\mathcal{R}_l(\mathbf{a}) = \begin{cases} \mathcal{R}_c(\mathbf{a}), & a_l \neq 0, \\ 0, & a_l = 0. \end{cases} \tag{7}$$

The relay searches for the best choice of the equation coefficient vector $\mathbf{a}$, in order to maximize the achievable rates. When the criterion is the maximization of $\mathcal{R}_c(\mathbf{a})$, the problem of finding the optimal $\mathbf{a}$ reduces to [7, Proposition 1]

$$\mathbf{a}_o = \arg\min_{\mathbf{a}\in\mathbb{Z}^2, \mathbf{a}\neq\mathbf{0}} \left(\mathbf{a}^T\mathbf{G}(\mathbf{h})\mathbf{a}\right), \tag{8}$$

where $\mathbf{G}(\mathbf{h}) = \mathbf{I} - \frac{P\left(\mathbf{h}\mathbf{h}^T\right)}{1+P\|\mathbf{h}\|^2}$.

Remark 1: In [9, Lemma 3.1], the authors propose the use of vectors without any zero elements. Indeed, when using only non-zero elements, their results show that the achievable transmission rates at the sources are increased for Gaussian channels. However, the overall system performance improvement depends on the statistics of the channel coefficients. For some channel statistics, the case where the best vector contains zero elements may be rare, so on average it may be favorable if one source does not transmit for an adverse channel realization, if the other can achieve a high data rate. Thus, in this work we include non-zero vectors with zero elements in the search for the optimal vector $\mathbf{a}_o$, but the proposed method can be easily modified to reject vectors with zero elements.

In [5, Lemma 1] it was also proven that

$$\|\mathbf{a}_o\|^2 \leq 1 + P\|\mathbf{h}\|^2. \tag{9}$$

Thus, the size of the set of candidate vectors is finite.

III. Offline Construction of Look-Up Tables

In this section we present an offline algorithm that is used to construct a set of joint look-up tables, based on which the relay will be able to choose a suboptimal equation coefficient vector $\mathbf{a}_{so}$, without performing the optimization in (8) for each channel instance. In the following we first present a method which reduces the number of candidate coefficient vectors, which consists of two steps:

- We present a theorem on the relation between the elements of the optimal vector, where we prove that the elements of the optimal equation coefficient vector are coprime.
- We then formulate a criterion for further reduction of the number of candidate equation coefficient vectors, by leveraging knowledge of the channel statistics and using power and accuracy specifications.

[1] Our method can be also applied to a complex system model after suitable modification, by decomposing into a real system model as in [5].

A. Reduction of the Number of Candidate Vectors

The number of candidate equation coefficient vectors is finite, due to (9). However, (9) is rather a loose upper bound for the norm of $\mathbf{a}_o$. Thus, in the following, we further restrict the set of equation coefficient vectors in which the optimal vector that satisfies (8) is contained.

Theorem 1: The optimal equation coefficient vector for the TWRC is either one of the vectors $[1,0]^T$, $[0,1]^T$, $[1,1]^T$, or its elements are coprime numbers.

Proof: Any vector with non-negative integer elements, except for $[1,0]^T$, $[0,1]^T$, $[1,1]^T$ and the vectors with coprime elements, can be written as a scaled version of these vectors. Let $\mathbf{a}$ be one of the vectors $[1,0]^T$, $[0,1]^T$, $[1,1]^T$, or a vector with coprime elements, and let $\bar{\mathbf{a}} = \lambda\mathbf{a}$ be a scaled version of $\mathbf{a}$, where $\lambda \in \mathbb{Z}^+$, $\lambda > 1$, is the greatest common divisor of the elements of $\bar{\mathbf{a}}$. Then

$$\bar{\mathbf{a}}^T\mathbf{G}\bar{\mathbf{a}} = \lambda^2\left(\mathbf{a}^T\mathbf{G}\mathbf{a}\right) > \mathbf{a}^T\mathbf{G}\mathbf{a}. \tag{10}$$

Since $\mathbf{a}_o$ minimizes the quantity in (8), $\mathbf{a}_o$ cannot be equal to $\bar{\mathbf{a}}$, but may be equal to $\mathbf{a}$. This concludes the proof. ■

All pairs of coprime numbers (m,n) with $m > n$ can be arranged in a pair of disjoint complete ternary trees, starting from $(2,1)$ or $(3,1)$ for even-odd or odd-odd pairs, respectively. The "children" of each vertex are generated as

$$(2m-n,m)\,,\quad (2m+n,m)\,,\quad (m+2n,n)\,. \tag{11}$$

This method is exhaustive and non-redundant with no invalid members [13], [14]. Furthermore, each child's norm is larger than the norm of the parent, so Theorem 1 combined with (9) and (11) can be used to form a finite set of candidate equation coefficient vectors, for a specific channel realization.

The number of candidate equation coefficient vectors can be reduced further by combining (9) with the statistical properties of the channel and the maximum SNR of interest.

We first introduce the instantaneous received SNR vector at the relay (corresponding to a specific $\mathbf{h}$) as

$$\mathbf{g} = [g_1, g_2]^T = [\sqrt{P}h_{\max}, \sqrt{P}h_{\min}]^T, \tag{12}$$

where $h_{\max} = \max(h_1,h_2)$ and $h_{\min} = \min(h_1,h_2)$. Note that the vectors $\mathbf{h}$ and $\mathbf{a}_o$ tend to be collinear [5]. Thus, we arrange the elements of $\mathbf{g}$ in descending order, so that the coprime pairs produced by (11), for which $m > n$, can be directly used as candidate coefficient vectors. Thus, if $\mathbf{a} = [a_1,a_2]^T$, a_1 will be used as an equation coefficient for the message of the source that corresponds to channel coefficient $h_{\max}$.

Let $P_{\max}$ be the maximum SNR of interest. We define the variable $\mathcal{S}(P_{\max},b)$ using

$$\Pr\left(\|\mathbf{g}\| \le \mathcal{S}(P_{\max},b)\right) = b, \tag{13}$$

where $\Pr(\cdot)$ denotes probability. The variable b denotes the probability with which the norm of $\mathbf{g}$ for $P = P_{\max}$ is smaller than $\mathcal{S}(P_{\max},b)$, e.g. 99% of all channel realizations. For Rayleigh distributed channel coefficients, the expression in (13) can be evaluated as follows. Note that

$$\|\mathbf{g}\|^2 = g_1^2 + g_2^2 = P\left(h_1^2+h_2^2\right), \tag{14}$$

is a random variable; using (2), $\|\mathbf{g}\|^2$ follows a scaled gamma distribution with pdf given by

$$f_{\|\mathbf{g}\|^2}(x) = \tfrac{1}{4P^2\sigma^4}xe^{-\frac{x}{2P\sigma^2}}, \quad x \ge 0 \tag{15}$$

and cumulative distribution function (cdf) given by

$$F_{\|\mathbf{g}\|^2}(x) = \gamma\left(2, \frac{x}{2P\sigma^2}\right), \tag{16}$$

where $\gamma(\cdot,\cdot)$ is the lower incomplete Gamma function [15]. Thus, $\mathcal{S}(P_{\max},b)$ is computed by setting $x = \mathcal{S}^2(P_{\max},b)$ and $P = P_{\max}$ and solving the equation

$$\gamma\left(2, \frac{\mathcal{S}^2(P_{\max},b)}{2P_{\max}\sigma^2}\right) = b. \tag{17}$$

The inverse of the lower incomplete Gamma function can be found via numerical optimization [16].

The equation coefficient vectors can now be bounded by (9), using $\mathcal{S}(P_{\max},b)$. Since (9) holds for a specific channel realization $\mathbf{h}$ and $P\|\mathbf{h}\|^2 = \|\mathbf{g}\|^2 \le \mathcal{S}^2(P_{\max},b)$ with probability b, using (11) all vectors $\mathbf{a}$ with coprime elements are constructed, for which it holds that

$$\|\mathbf{a}\|^2 \le 1 + \mathcal{S}^2(P_{\max},b). \tag{18}$$

B. Offline Algorithm for the Construction of Look-Up Tables

However, even after using (18), the number of candidate equation coefficient vectors is still large. In the following we elaborate on how the look-up tables are constructed. The construction of the tables is done in a way that not only enables low-complexity search during the operation of the system, but also reduces the number of candidate vectors even further.

The optimization in (8) tends to choose a vector $\mathbf{a}$ that is almost collinear with $\mathbf{h}$, but also has a norm that is as small as possible. Indeed, expanding (8) and after some algebraic manipulation, the problem is equivalent to

$$\begin{aligned}\mathbf{a}_o &= \arg\min_{\mathbf{a}\in\mathbb{Z}^2,\mathbf{a}\neq\mathbf{0}}\left(\|\mathbf{a}\|^2 + P\left(\|\mathbf{a}\|^2\|\mathbf{h}\|^2 - (\mathbf{h}^T\mathbf{a})^2\right)\right)\\ &= \arg\min_{\mathbf{a}\in\mathbb{Z}^2,\mathbf{a}\neq\mathbf{0}}\left(\|\mathbf{a}\|^2 + P\|\mathbf{a}\|^2\|\mathbf{h}\|^2\sin^2\left(\angle(\mathbf{a},\mathbf{h})\right)\right),\end{aligned} \tag{19}$$

where $\angle(\cdot,\cdot)$ denotes the angle between two vectors. Hence, for large P, the vectors that are selected will tend to be collinear to $\mathbf{h}$. On the other hand, for small values of P, a vector with larger $\angle(\mathbf{a},\mathbf{h})$ but smaller norm may be chosen instead. Assume that $\mathbf{h}$ happens to be the same as some candidate $\mathbf{a}$, i.e., $\mathbf{h} = \mathbf{a}$. Then if there exists a value $P_{\min}(\mathbf{a})$ below which the solution of (8) is not $\mathbf{a}$ (because its norm is too large compared to $\mathbf{g}$) $\mathbf{a}$ will never be chosen when $P \le P_{\min}(\mathbf{a})$ whatever $\mathbf{h}$ may be, since the second term of (19) will become nonzero when $\mathbf{h}$ and $\mathbf{a}$ are not collinear. Hence, if $\|\mathbf{g}\|^2$ is smaller than $P_{\min}(\mathbf{a})\|\mathbf{a}\|^2$, we know that $\mathbf{a}$ cannot be a solution of (8) and can therefore be excluded from the search space. For all the $\mathbf{a}$ that safisfy Theorem 1 and (18) $P_{\min}(\mathbf{a})$ is the solution of the following problem

$$\begin{aligned} \textbf{minimize} \quad & P \\ \textbf{subject to} \quad & \arg\min_{\mathbf{x}\neq\mathbf{0}} \left(\mathbf{x}^T \mathbf{G}\left(\mathbf{a}\right)\mathbf{x}\right) = \mathbf{a} \text{ and} \\ & \mathbf{G}\left(\mathbf{a}\right) = \mathbf{I} - \tfrac{P\left(\mathbf{a}\mathbf{a}^T\right)}{1+P\|\mathbf{a}\|^2}. \end{aligned} \tag{20}$$

Thus, if

$$R(\mathbf{a}) \triangleq \sqrt{P_{\min}(\mathbf{a})\|\mathbf{a}\|^2} \leq \mathcal{S}(P_{\max}, b), \tag{21}$$

then $\mathbf{a}$ will be considered as a candidate vector, else it is discarded. Moreover, for a given $\mathbf{g}$, $\mathbf{a}$ is a candidate vector only when $\|\mathbf{g}\| > R(\mathbf{a})$.

Therefore, the $R(\mathbf{a})$ can be used to partition the two-dimensional space of (g_1, g_2) into rings. For each channel instance, the search algorithm only considers the candidate vectors that correspond to the largest ring such that $\|\mathbf{g}\| > R(\mathbf{a})$. The proposed offline algorithm constructs a vector $\mathbf{r}$ of length L, containing all values $R(\mathbf{a}) < \mathcal{S}(P_{\max}, b)$ with ascending order, and a corresponding matrix $\mathbf{A}$, whose columns are the candidate vectors that remain after applying (21), sorted so that $r_j = R(\mathbf{A}(j))$, where $\mathbf{A}(j)$ is the j-th column of $\mathbf{A}$. Thus, vector $\mathbf{r}$ is used to partition the space of $\mathbf{g}$ into rings, whereas the vectors in $\mathbf{A}$ are sorted in order of appearance as candidate vectors, as $\|\mathbf{g}\|$ grows. Note that $r_1 = 0$.

As a final step in reducing the complexity of the search, we return to (19) and observe that candidate vectors that form a small angle with the channel are more likely to be the optimal solution. Therefore, the search algorithm also makes use of the angle of $\mathbf{g}$ by partitioning the search space into sectors and choosing the candidate vector corresponding to the sector determined by the angle of $\mathbf{g}$. For the vectors in $\mathbf{A}$, the phase angle $\phi(\mathbf{A}(j)) = \angle(\mathbf{A}(j))$ is computed. The algorithm constructs the matrices $\mathbf{A}^j$, $j = 1, 2, \ldots, L$, each containing the first j vectors of $\mathbf{A}$, but in ascending order with respect to $\phi(\mathbf{A}(k))$, $k = 1, 2, \ldots, j$. Finally, for $j \geq 2$, the algorithm constructs the vectors $\boldsymbol{\theta}^j$, which contain the boundaries θ_k^j that form the sectors in each ring. $\theta_1^j = 0$, while for $k = 2, 3, \ldots, j$, the boundaries θ_k^j are computed by

$$\theta_k^j = \frac{|\mathbf{A}^j(k-1)|\phi(\mathbf{A}^j(k-1)) + |\mathbf{A}^j(k)|\phi(\mathbf{A}^j(k))}{|\mathbf{A}^j(k-1)| + |\mathbf{A}^j(k)|}. \tag{22}$$

The above choice for θ_k^j reflects the fact that a vector with smaller norm is more likely to be chosen as seen in (19), compared to a vector with larger norm, when their phase angles are not very different. Thus, the sectors within a ring that correspond to vectors with large norm are narrower than those corresponding to vectors with small norm.

The algorithm is summarized in Algorithm 1.

IV. Selection of the Equation Coefficient Vector

The relay selects an equation coefficient vector $\mathbf{a}_{so}$, using the look-up tables constructed by Algorithm 1. The selection is performed based on the channel coefficients, and is achieved by performing simple comparisons.

Algorithm 1 Offline construction of look-up tables

procedure CONSTRUCTION($\mathcal{S}(P_{\max}, b)$)
- Set, $\mathbf{a}_1 \leftarrow [1, 0]^T$, $\mathbf{a}_2 \leftarrow [1, 1]^T$, $\mathbf{a}_3 \leftarrow [2, 1]^T$, $\mathbf{a}_4 \leftarrow [3, 1]^T$.
- Create all vectors $\mathbf{a}_i$, $i \geq 5$ for which $\|\mathbf{a}_i\|^2 \leq 1 + \mathcal{S}^2(P_{\max}, b)$, using $\mathbf{a}_3$, $\mathbf{a}_4$ and (11). Let $i_{\max}$ be the number of the resulting vectors.
- **for** $i = 1 : i_{\max}$ **do**
  - $P_{\min}(\mathbf{a}_i) \leftarrow$ Solution of (20) for $\mathbf{a} = \mathbf{a}_i$.
  - $R(\mathbf{a}_i) \leftarrow \sqrt{P_{\min}(\mathbf{a}_i)\|\mathbf{a}_i\|^2}$.
  - **if** $R(\mathbf{a}_i) > \mathcal{S}(P_{\max}, b)$ **then**
    - Discard $\mathbf{a}_i$.
  - **end if**
- **end for**
- Create vector $\mathbf{r}$ containing the values $R(\mathbf{a}_i)$ for the non-discarded vectors with ascending order.
- $L \leftarrow \text{length}(\mathbf{r})$.
- Create matrix $\mathbf{A}$ with L columns denoted by $\mathbf{A}(j)$, which contains the non-discarded vectors, so that $r_j = R(\mathbf{A}(j))$, where $\mathbf{A}(j)$ is the j-th column of $\mathbf{A}$.
- Compute the phase angles $\phi(\mathbf{A}(j)) = \arctan\left(\frac{A(2,j)}{A(1,j)}\right)$, where $A(i, j)$ are the elements of $\mathbf{A}$.
- **for** j=1:L **do**
  - Create matrix $\mathbf{A}^j$, which contains the first j columns of $\mathbf{A}$, permuted so that $\phi\left(\mathbf{A}^j(k)\right) \leq \phi\left(\mathbf{A}^j(k+1)\right)$, $\forall k = 1, 2, \ldots j$.
  - **if** $j \neq 1$ **then**
    - Create vector $\boldsymbol{\theta}^j$ of length j, whose elements are computed according to Eq. (22).
  - **end if**
- **end for**
- **return** $\mathbf{r}$, $\mathbf{A}^j$ for $j = 1, 2, \ldots L$, and $\boldsymbol{\theta}^j$ for $j = 2, 3, \ldots L$.

end procedure

More specifically, the relay computes the norm and phase angle of the vector $\mathbf{g}$ that can be found using the channel state information (CSI) available at the relay. The phase angle is given by

$$t = \arctan\left(\frac{g_2}{g_1}\right). \tag{23}$$

The norm $\|\mathbf{g}\|$ is compared with the elements r_j of vector $\mathbf{r}$, and the ring for which $r_j \leq \|\mathbf{g}\| < r_{j+1}$ is selected. For this value of j, the phase angle of $\mathbf{g}$ is compared with the boundaries in $\boldsymbol{\theta}^j$, and sector k is selected that satisfies $\theta_k^j \leq t < \theta_{k+1}^j$. The equation coefficient vector that is chosen is $\mathbf{A}^j(k)$. Finally, the elements of $\mathbf{A}^j(k)$ may need to be permuted based on whether $h_{\max} = h_1$ or $h_{\max} = h_2$.

Note that the complexity of a search in a sorted vector is $O(\log L)$, where L is the size of the vector. Thus, since the number of candidate equation coefficient vectors is at most L (the size of $\mathbf{r}$), it is clear that the computations performed at the relay for the selection of an equation coefficient vector are of very low complexity.

The selection Algorithm is summarized in Algorithm 2.

V. Results and Discussion

In this section the proposed methodology for the selection of the equation coefficients is applied to a specific case. The channel coefficients follow a Rayleigh distribution with $\Omega =$

Algorithm 2 Selection algorithm for $\mathbf{a}_{so}$

```
procedure SELECTION(h, P, r, A^j, θ^j, ∀j)
    Compute g using (12).
    Compute t using (23).
    Find j ∈ {1, 2, ..., L} such that r_j ≤ ||g|| < r_{j+1}. When ||g|| ≥ r_L, j = L.
    if j = 1 then
        Select A^1.
    else
        Find k ∈ 1, 2, ..., j such that θ^j_k ≤ t < θ^j_{k+1}. When t ≥ θ^j_j, k = j.
        Select A^j(k).
    end if
    if h_1 > h_2 then
        a_so ← [A^j(1,k), A^j(2,k)]^T.
    else
        a_so ← [A^j(2,k), A^j(1,k)]^T.
    end if
    return a_so
end procedure
```

$2\sigma^2 = 1$, the maximum SNR value of interest is $P_{\max} = 30$ dB and the percentage of interest for the channel realizations is $b = 99\%$. Using (17) for $\sigma = \frac{1}{\sqrt{2}}$, $\mathcal{S}(30\,\text{dB}, 0.99) = 81.4761$.

During the execution of the offline construction algorithm, the number of candidate vectors is substantially reduced. Specifically, although the remaining vectors are $i_{\max} = 1590$ after line 3 of Algorithm 1 is executed, after lines $4-10$, this number is drastically reduced to $L = 21$, which are put into the matrix $\mathbf{A}$, constructed in line 13. This means that, for the selected maximum value of SNR and for the 99% of channel realizations, it suffices to search only among 21 vectors. Moreover, note that if $\|\mathbf{g}\| \leq \mathcal{S}(P_{\max}, b)$, the optimal solution $\mathbf{a}_o$ of (8) is contained in these 21 vectors. Furthermore, for small values of the SNR, the number of candidate vectors is smaller than L, since the relay searches only among the vectors contained in the selected matrix $\mathbf{A}^j$.

The set of look-up tables constructed by Algorithm 1 can be graphically illustrated as a partition of the two-dimensional space of g_1, g_2, as shown in Fig. 1. The space is partitioned into rings and sectors, and each partition is mapped to a specific equation coefficient vector. Only the angles in the interval $\left[0, \frac{\pi}{4}\right]$ rad need to be considered, since it always holds that $g_1 > g_2$. Thus, for each specific channel realization, when the vector $\mathbf{g}$ falls in a specific ring and sector, the relay directly selects $\mathbf{a}_{so}$, executing Algorithm 2.

The equation coefficient vectors used in the look-up tables for the specific case that is considered in this section are given in Table I. Only the first i vectors are mapped in the i-th ring. The mapping between rings and sectors and the equation coefficient vectors is given in Table II. Each line corresponds to a ring in Fig. 1, with ascending order of radius. In each line, the indices of vectors are as given in Table I, appearing in correspondence with each consequent sector, as illustrated in Fig. 1, with ascending order of angle. It is easily seen that the first and the last sector are always mapped to the vectors $\mathbf{a}_1 = [1, 0]^T$ and $\mathbf{a}_2 = [1, 1]^T$, respectively. This is expected since in first sector $g_2 \simeq 0$, whereas in the last sector $g_1 \simeq g_2$.

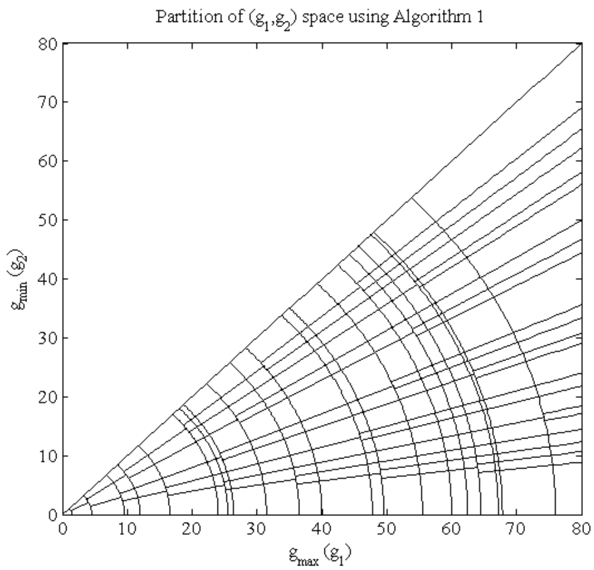

Fig. 1. Partition of the two-dimensional space of vector $\mathbf{g}$ into rings and sectors, which are mapped to specific equation coefficient vectors.

TABLE I
CANDIDATE VECTORS

Number	Vector	Number	Vector	Number	Vector
$\mathbf{a}_1$	$[1,\ 0]^T$	$\mathbf{a}_8$	$[5,\ 1]^T$	$\mathbf{a}_{15}$	$[7,\ 3]^T$
$\mathbf{a}_2$	$[1,\ 1]^T$	$\mathbf{a}_9$	$[5,\ 2]^T$	$\mathbf{a}_{16}$	$[6,\ 5]^T$
$\mathbf{a}_3$	$[2,\ 1]^T$	$\mathbf{a}_{10}$	$[5,\ 3]^T$	$\mathbf{a}_{17}$	$[7,\ 4]^T$
$\mathbf{a}_4$	$[3,\ 1]^T$	$\mathbf{a}_{11}$	$[6,\ 1]^T$	$\mathbf{a}_{18}$	$[8,\ 1]^T$
$\mathbf{a}_5$	$[3,\ 2]^T$	$\mathbf{a}_{12}$	$[5,\ 4]^T$	$\mathbf{a}_{19}$	$[7,\ 5]^T$
$\mathbf{a}_6$	$[4,\ 1]^T$	$\mathbf{a}_{13}$	$[7,\ 2]^T$	$\mathbf{a}_{20}$	$[8,\ 3]^T$
$\mathbf{a}_7$	$[4,\ 3]^T$	$\mathbf{a}_{14}$	$[7,\ 1]^T$	$\mathbf{a}_{21}$	$[9,\ 2]^T$

In Fig. 2, the results for the computation rate of the relay, when the proposed procedure is performed, are compared to the computation rate achieved when the optimization in (8) is performed for each channel realization. It is evident that the proposed method leads to a result that is very close to the optimal computation rate. Specifically, the proposed method achieves rates that are less than 0.3 dB below the optimal rates, for the whole range of SNR values of interest.

Finally, the average number of candidate vectors that are searched is presented in Table III for various SNR values, as computed via simulation. Since the worst-case scenario corresponds to a search among 21 vectors, when $\mathbf{g}$ falls in a ring of smaller radius, the average search is among even fewer vectors, as is clearly indicated. Furthermore, the search is performed using simple comparisons and does not require the solution of an optimization problem, which leads to very low complexity.

TABLE II
MAPPING BETWEEN SECTORS AND CANDIDATE VECTORS

Interval for $\|\mathbf{g}\|$	Index of vector corresponding to each sector with ascending order of angle
(r_1, r_2)	1
(r_2, r_3)	$1, 2$
(r_3, r_4)	$1, 3, 2$
(r_4, r_5)	$1, 4, 3, 2$
(r_5, r_6)	$1, 4, 3, 5, 2$
(r_6, r_7)	$1, 6, 4, 3, 5, 2$
(r_7, r_8)	$1, 6, 4, 3, 5, 7, 2$
(r_8, r_9)	$1, 8, 6, 4, 3, 5, 7, 2$
(r_9, r_{10})	$1, 8, 6, 4, 9, 3, 5, 7, 2$
(r_{10}, r_{11})	$1, 8, 6, 4, 9, 3, 10, 5, 7, 2$
(r_{11}, r_{12})	$1, 11, 8, 6, 4, 9, 3, 10, 5, 7, 2$
(r_{12}, r_{13})	$1, 11, 8, 6, 4, 9, 3, 10, 5, 7, 12, 2$
(r_{13}, r_{14})	$1, 11, 8, 6, 13, 4, 9, 3, 10, 5, 7, 12, 2$
(r_{14}, r_{15})	$1, 14, 11, 8, 6, 13, 4, 9, 3, 10, 5, 7, 12, 2$
(r_{15}, r_{16})	$1, 14, 11, 8, 6, 13, 4, 9, 15, 3, 10, 5, 7, 12, 2$
(r_{16}, r_{17})	$1, 14, 11, 8, 6, 13, 4, 9, 15, 3, 10, 5, 7, 12, 16, 2$
(r_{17}, r_{18})	$1, 14, 11, 8, 6, 13, 4, 9, 15, 3, 17, 10, 5, 7, 12, 16, 2$
(r_{18}, r_{19})	$1, 18, 14, 11, 8, 6, 13, 4, 9, 15, 3, 17, 10, 5, 7, 12, 16, 2$
(r_{19}, r_{20})	$1, 18, 14, 11, 8, 6, 13, 4, 9, 15, 3, 17, 10, 5, 19, 7, 12, 16, 2$
(r_{20}, r_{21})	$1, 18, 14, 11, 8, 6, 13, 4, 20, 9, 15, 3, 17, 10, 5, 19, 7, 12, 16, 2$
(r_{21}, ∞)	$1, 18, 14, 11, 8, 21, 6, 13, 4, 20, 9, 15, 3, 17, 10, 5, 19, 7, 12, 16, 2$

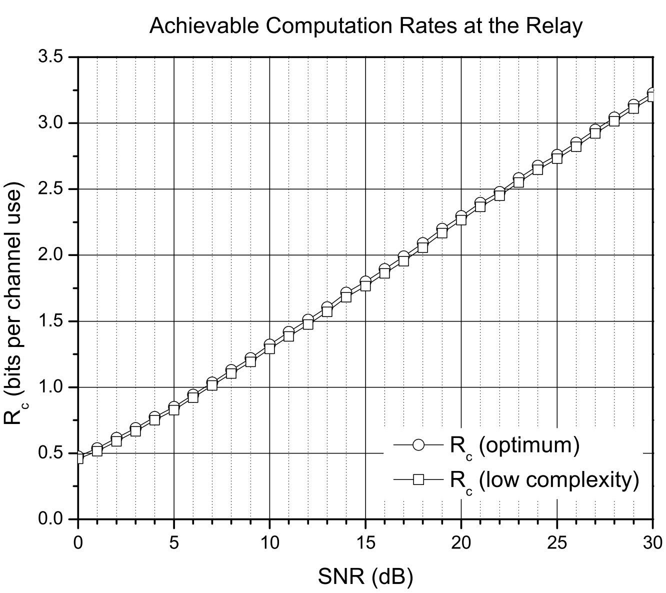

Fig. 2. Optimal and low-complexity computation of rate R_c at the relay.

VI. CONCLUSIONS

In this paper, a low-complexity method was presented that uses look-up tables to perform PHY-layer network coding in two-way compute-and-forward relaying channels. An offline algorithm is used to construct the look-up tables, which reduces the number of candidate equation coefficient vectors. Moreover, a second low-complexity algorithm was presented for the selection of the vector at the relay. The proposed method achieves near optimal performance in terms of computation rate with very low computational cost. The proposed algorithms can be efficiently used in the uplink of future wireless communication networks.

TABLE III
AVERAGE NUMBER OF CANDIDATE VECTORS

SNR [dB]	Average number of candidate vectors	SNR [dB]	Average number of candidate vectors
0	1.3461	20	4.6150
5	1.8630	25	7.1785
10	2.3831	30	11.6160
15	3.1459		

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