Properties of the EGC Output SNR over Correlated Generalized-Fading Channels

Dimitris A. Zogas, Stavros A. Kotsopoulos
Electrical & Computer Engineering Dept.
University of Patras
Rion, 26442 Patras, Greece
zogas@space.noa.gr, kotsop@ee.upatras.gr

George K. Karagiannidis, P. Takis Mathiopoulos*
Institute for Space Applications & Remote Sensing
National Observatory of Athens
Palea Penteli, 15236 Athens, Greece
gkarag@space.noa.gr, mathio@space.noa.gr
* Also with the Department of Electrical & Computer Engineering
University of British Columbia, Vancouver, Canada

***Abstract*— In this paper, important statistical parameters, such as the *k*-moment and the *k*-central moment, of the output signal-to-noise ratio (SNR), in predetection equal-gain combining (EGC) systems operating over correlated Nakagami-*m* fading channels are studied. Simple closed-form expressions for the mean and variance of the output SNR are presented. Furthermore, it is shown that our general results reduce to the specific non-correlative fading case previously published. Moreover, significant performance criteria for the EGC such as amount of fading (AoF) and spectral efficiency in the low power regime are investigated. Numerical and simulation results are presented validating the proposed mathematical analysis. These results also point out the impact of the fading correlation, the input SNRs unbalancing as well as the fading severity, on the EGC performance.**

Keywords: *Amount of fading, antenna diversity, average output SNR, correlative fading, equal-gain combining (EGC), Nakagami-m fading channels, moments, spectral efficiency.*

I. INTRODUCTION

Antenna diversity can be efficiently used to reduce the effects of fading and to improve the performance of wireless communications systems. Various techniques are used to combine the signals from multiple diversity branches. The most popular of them are maximal-ratio combining (MRC), equal-gain combining (EGC), selection combining (SC) and switch combining (SWC) [1]. EGC is of great practical interest because it provides an intermediate solution as far as the performance and the implementation complexity are concerned.

A comprehensive summary with most of the related work on EGC with independent fading is included in [1]. However, independent fading is not always realized in practice due to insufficient antenna spacing, when diversity is applied in small terminals. Therefore, it is important to understand how the correlation between received signals affects the offered diversity gain. From reviewing the literature there are few approaches for the performance evaluation of predetection EGC over correlated fading channels. In [2] a formula for the error probability of EGC with orthogonal binary frequency-shift-keying (BFSK) operating in correlative Rician time-selective fading is proposed, while in [3] the BER performance of dual MRC, EGC and SC over correlated Rayleigh channels is studied. Recently, Mallik *et al.* in [4] presented a useful approach analytically evaluating the BER performance of binary signals with coherent detection employing dual predetection EGC over correlated Rayleigh channels. Nevertheless, although in general BER is good indication of the system's performance, it does not provide sufficient information about the type of errors occurring. For example, BER does not give any indication of bursty errors. Other performance criteria include probability of outage, amount of fading (AoF) [1, p.18] and spectral efficiency. However, it appears that the most common and best-understood performance criterion of a digital communication system is the SNR typically measured at the output of the receiver. The reason for this choice is because it is the easiest to evaluate SNR and also it serves as an excellent indicator of the systems fidelity [1]. In previous related works, average combined SNR is used to evaluate diversity systems operating over independent fading channels employing SC-MRC-EGC [1] or over correlated fading channels with SC [5], [6].

In this paper, closed-form expressions for important statistical parameters such as the *k*-moment and the *k*-central moment of the output SNR, in predetection EGC systems operating over correlated Nakagami-*m* fading channels are derived. This approach can be efficiently used to overcome the infeasible derivation of the pdf of the output SNR, by evaluating parameters as the mean and the variance that characterize the behavior of the distribution. Furthermore, significant performance criteria such as the AoF and the spectral efficiency in the low power regime of the EGC are studied. Finally, simulations are performed to check the accuracy of the proposed mathematical analysis.

II. PROPERTIES OF THE EGC OUTPUT SNR

In a predetection EGC receiver all L diversity signals together with the Additive White Gaussian Noise (AWGN) are co-phased and multiplied by the same branch gain g, before being summed to give the resultant output signal. It is assumed that the transmitted signal is received over correlated slowly varying Nakagami-*m* flat fading channels [7] and that the AWGN is statistically independent from branch to branch with a power spectral density N_0, equal for all paths and also

0-7803-7974-8/03/$17.00

independent of the correlated Nakagami-m fading amplitudes $a_1,\dots,a_L$. The output SNR of the L branches EGC receiver is [1, (9.46)].

$$\gamma_{out,L} = \left(a_1 + \ldots + a_L\right)^2 E_s / \left(L N_0\right) \tag{1}$$

were E_s is the energy per symbol.

A. Moments

By definition, the k-moment of the output SNR can be written as [8, (2.80)]

$$\begin{aligned} E\left\langle \gamma_{out,L}^k \right\rangle &= E\left\langle \left[\frac{E_s \left(a_1 + \ldots + a_L\right)^2}{L N_0} \right]^k \right\rangle \\ &= \left(\frac{E_s}{L N_0} \right)^k E\left\langle \left(a_1 + \ldots + a_L\right)^{2k} \right\rangle \end{aligned} \tag{2}$$

where $E\langle . \rangle$ means expectation. Expanding $\left(a_1 + \ldots + a_L\right)^{2k}$ with the use of the multinomial identity [9, (24.1.2)], (2) can be written as

$$E\left\langle \gamma_{out,L}^k \right\rangle = \begin{cases} 1, & \text{for } k = 0 \\ \left(\dfrac{E_s}{L N_0} \right)^k \left[\displaystyle\sum_{i=1}^{L} E\left\langle a_i^{2k} \right\rangle + \sum_{\substack{n_1,\ldots,n_L=0 \\ n_1+\ldots+n_L=2k}}^{2k} \frac{(2k)!}{n_1!\ldots n_L!} \right. & \\ \left. \times E\left\langle a_1^{n_1} \ldots a_L^{n_L} \right\rangle \displaystyle\prod_{j=1}^{L} \left[1 - \delta\left(n_j, 2k\right)\right] \right], & \text{for } k > 0 \end{cases} \tag{3}$$

where $\delta(\cdot)$ is the well-known delta function defined as

$$\delta(i,j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{4}$$

or

$$E\left\langle \gamma_{out,L}^k \right\rangle = \begin{cases} 1, & \text{for } k = 0 \\ \dfrac{1}{L^k} \displaystyle\sum_{i=1}^{L} E\left\langle \gamma_i^k \right\rangle + \left(\frac{E_s}{L N_0} \right)^k \sum_{\substack{n_1,\ldots,n_L=0 \\ n_1+\ldots+n_L=2k}}^{2k} \left[\frac{(2k)!}{n_1!\ldots n_L!} \right. & \\ \left. \times E\left\langle a_1^{n_1} \ldots a_L^{n_L} \right\rangle \displaystyle\prod_{j=1}^{L} \left[1 - \delta\left(n_j, 2k\right)\right] \right], & \text{for } k > 0 \end{cases} \tag{5}$$

with $\gamma_i = E_s a_i^2 / N_0$, $i = 1,\ldots,L$ being the instantaneous SNR at the i^{th} input branch of the combiner. Using the expression for the k-moment of the SNR of a single Nakagami-m channel [1, (2.23)] (5) can be written as

$$E\left\langle \gamma_{out,L}^k \right\rangle = \begin{cases} 1, & \text{for } k = 0 \\ \dfrac{\Gamma(m+k)}{L^k \Gamma(m) m^k} \displaystyle\sum_{i=1}^{L} \left(\overline{\gamma}_i\right)^k + \left(\frac{E_s}{L N_0} \right)^k \sum_{\substack{n_1,\ldots,n_L=0 \\ n_1+\ldots+n_L=2k}}^{2k} \left[\frac{(2k)!}{n_1!\ldots n_L!} \right. & \\ \left. \times E\left\langle a_1^{n_1} \ldots a_L^{n_L} \right\rangle \displaystyle\prod_{j=1}^{L} \left[1 - \delta\left(n_j, 2k\right)\right] \right], & \text{for } k > 0 \end{cases} \tag{6}$$

with $\overline{\gamma}_i$ being the average SNR of the i^{th} input branch.

Using the above equation, $E\left\langle \gamma_{out,L}^k \right\rangle$ will be evaluated for the special case of $k = 1$, which corresponds to the important statistical parameter of the average output SNR and then it will be generalized for arbitrary values of k. For $k = 1$, it is easily recognized that only terms in the form $E\left\langle a_i a_j \right\rangle$ appear in (6). These terms are defined as [8]

$$E\left\langle a_i a_j \right\rangle = \int_0^\infty \int_0^\infty a_i a_j f_{a_i,a_j}\left(a_i, a_j\right) da_i\, da_j \tag{7}$$

with $f_{a_i,a_j}\left(a_i, a_j\right)$ being the well-known bivariate Nakagami-m pdf expressed as [7]

$$\begin{aligned} f_{a_i,a_j}\left(a_i, a_j\right) = \frac{4 m^{m+1} \left(a_i a_j\right)^m e^{-\frac{m}{1-\rho_{i,j}} \left(\frac{a_i^2}{\Omega_i} + \frac{a_j^2}{\Omega_j} \right)}}{\Gamma(m) \Omega_i \Omega_j \left(1 - \rho_{i,j}\right) \left(\sqrt{\Omega_i \Omega_j \rho_{i,j}} \right)^{m-1}} \\ \times I_{m-1}\left(\frac{2m \sqrt{\rho_{i,j}}\, a_i a_j}{\sqrt{\Omega_i \Omega_j} \left(1 - \rho_{i,j}\right)} \right) \end{aligned} \tag{8}$$

where $\Omega_i = \overline{a_i^2}$, with $\overline{a_i^2}$ being the average signal power of the i^{th} branch (the 2^{nd} moment of a_i), $I_v(\cdot)$ is the first kind and v^{th} order modified Bessel function and $\rho_{i,j}$ is the power correlation coefficient. The double integral in (7) can be expressed in a closed form as [7, (137)]

$$E\left\langle a_i a_j \right\rangle = \frac{\sqrt{\Omega_i \Omega_j} \left[\Gamma\left(m + \frac{1}{2} \right) \right]^2 {}_2F_1\left(-\frac{1}{2}, -\frac{1}{2}; m; \rho_{i,j} \right)}{m \left[\Gamma(m) \right]^2} \tag{9}$$

with ${}_2F_1\left(x_1, x_2, y_1, z\right)$ being the Gauss hyper-geometric function [10, (9.100)], available in most of the well-known mathematical software packets, as MATHEMATICA, MAPLE, etc. Using (6) and, (9) and after some algebraic manipulations, the average output SNR (1^{st} moment) of a EGC receiver with L correlated branches can be expressed as

$$\bar{\gamma}_{out,L} = \frac{1}{L}\sum_{l=1}^{L}\bar{\gamma}_l + \frac{\left[\Gamma\left(m+\frac{1}{2}\right)\right]^2}{L\,m\left[\Gamma(m)\right]^2}\sum_{i,j=1}^{L}\sqrt{\bar{\gamma}_i\,\bar{\gamma}_j} \times {}_2F_1\left(-\frac{1}{2},-\frac{1}{2};m;\rho_{i,j}\right)\left[1-\delta(i,j)\right]. \quad (10)$$

Equation (10) presents a novel general but yet simple closed-form expression for obtaining the output SNR for arbitrary number of branches L, arbitrary values of the fading severity parameter m, unequal branch powers and arbitrary correlation of the diversity paths. Moreover, if these paths are uncorrelated (i.e. $\rho_{i,j}=0$ for every i, j) and identically distributed (i.e. $\bar{\gamma}_i=\bar{\gamma},\ i=1,\ldots,L$) then (10) reduces to the well-known published formula for the average output SNR of an EGC receiver with L independent branches [1, (9.48)]:

$$\bar{\gamma}_{out,L\_Ind} = \bar{\gamma}\left(1+(L-1)\frac{\left[\Gamma\left(m+\frac{1}{2}\right)\right]^2}{m\left[\Gamma(m)\right]^2}\right). \quad (11)$$

Now, in order to generalize our results for $k>1$ it is necessary to derive a closed form expression for

$$E\left\langle a_1^{n_1}\ldots a_L^{n_L}\right\rangle = \int_0^\infty\ldots\int_0^\infty a_1^{n_1}\ldots a_L^{n_L}\, f_{a_1,\ldots,a_L}(a_1,\ldots,a_L)\,da_1\ldots da_L \quad (12)$$

with $f_{a_1,\ldots,a_L}(a_1,\ldots,a_L)$ being the multivariate (joint) Nakagami-m distribution of $a_1,\ldots,a_L$. Unfortunately, a known formula for $f_{a_1,\ldots,a_L}(a_1,\ldots,a_L)$ when $L>2$, does not exist in the open technical literature. However, recently the authors in [11], [12] presented an efficient approximation to the multivariate Nakagami-m distribution with arbitrary correlation. According to this approach $f_{a_1,\ldots,a_L}(a_1,\ldots,a_L)$ can be mathematically expressed as

$$f_{a_1,\ldots,a_L}(a_1,\ldots,a_L) = \frac{|\mathbf{W}|^m\, a_1^{m-1}\, a_L^m\, e^{-\sum_{n=1}^{L} w_{n,n} a_n^2/2}}{2^{m-1}\Gamma(m)} \times \prod_{n=1}^{L-1}\left[\frac{a_n\, I_{m-1}\left(\left|w_{n,n+1}\right| a_n\, a_{n+1}\right)}{\left|w_{n,n+1}\right|^{(m-1)}}\right] \quad (13)$$

where $\mathbf{W}$ is the inverse of the correlation matrix $\mathbf{\Sigma}$, i.e. $\mathbf{W}=\mathbf{\Sigma}^{-1}$ with elements $w_{i,j}$, $1\le i,j\le L$. For the special case of exponentially correlated fading, i.e. $\mathbf{\Sigma}_{i,j}\equiv\rho^{|i-j|}$, (13) can be used directly to evaluate $f_{a_1,\ldots,a_L}(a_1,\ldots,a_L)$. If $\mathbf{\Sigma}$ is an arbitrary correlation matrix, then its entries must be approximated with the elements of a Green's matrix [13], $\mathbf{C}$, in order $\mathbf{W}=\mathbf{C}^{-1}$ being tridiagonal. Without loss of generality and for simplification purposes it is assumed in (13) that $\Omega_i=2\sigma_i^2$, with $\sigma_i^2=1$ being the variance of the input signal at the i^{th} branch. Following the same procedure as in [7] for the evaluation of $E\left\langle a_1^{n_1}a_2^{n_2}\right\rangle$, $E\left\langle a_1^{n_1}\ldots a_L^{n_L}\right\rangle$ can be expressed as

$$\begin{aligned}E\left\langle a_1^{n_1}\ldots a_L^{n_L}\right\rangle = \frac{|\mathbf{W}|^m}{2^{2Lm+\sum_{j=1}^{L-1}\frac{n_j}{2}}}\sum_{i_1,\ldots,i_{L-1}=0}^{\infty}\Bigg\{&\left[\prod_{j=1}^{L-1}\frac{\left|w_{j,j+1}\right|^{2i_j}}{2^{4i_j}\, i_j!\,\Gamma\left(m+i_j\right)}\right]\\ &\times w_{1,1}^{m+i_1+n_1/2}\, w_{2,2}^{m+i_1+i_2+n_2/2}\ldots w_{L-1,L-1}^{m+i_{L-2}+i_{L-1}+n_{L-1}/2}\, w_{L,L}^{m+i_{L-1}+n_L/2}\\ &\times\Gamma\left(m+i_1+\frac{n_1}{2}\right)\Gamma\left(m+i_1+i_2+\frac{n_2}{2}\right)\\ &\times\ldots\Gamma\left(m+i_{L-2}+i_{L-1}+\frac{n_{L-1}}{2}\right)\Gamma\left(m+i_{L-1}+\frac{n_L}{2}\right)\Bigg\}\end{aligned} \quad (14)$$

where $|\mathbf{W}|$ denotes the norm of $\mathbf{W}$. Hence, using (6) and (14) the k-moment of the output SNR can be evaluated in its general form. The infinite series in (14) have a fast convergence and a mean number of 10 terms for each sum are sufficient for accuracy at the 5^{th} significant figure assuming exponential correlation between the branches and signals envelopes ranging from 1 - 10 [11].

B. Central Moments

The k-central moment of the output SNR, $E\left\langle\left(\gamma_{out,L}-\bar{\gamma}_{out,L}\right)^k\right\rangle$, can be written after using the binomial identity [9, (3.1.1)], as

$$\begin{aligned}E\left\langle\left(\gamma_{out,L}-\bar{\gamma}_{out,L}\right)^k\right\rangle &= \sum_{n=0}^{k}\frac{k!}{n!(k-n)!}E\left\langle\gamma_{out,L}^n\left(-\bar{\gamma}_{out,L}\right)^{k-n}\right\rangle\\ &= \sum_{n=0}^{k}\frac{k!(-1)^{k-n}\left(\bar{\gamma}_{out,L}\right)^{k-n}}{n!(k-n)!}E\left\langle\gamma_{out,L}^n\right\rangle\end{aligned} \quad (15)$$

where $E\left\langle\gamma_{out,L}^n\right\rangle$ is the n^{th} moment of $\gamma_{out,L}$ given by (6). Setting $k=2$ in (15), another important statistical parameter, the variance of the output SNR, can be derived as

$$\operatorname{var}\left\langle\gamma_{out,L}\right\rangle = E\left\langle\gamma_{out,L}^2\right\rangle-\left(\bar{\gamma}_{out,L}\right)^2 \quad (16)$$

where $E\left\langle\gamma_{out,L}^2\right\rangle$ is the 2^{nd} moment of $\gamma_{out,L}$, evaluated using (6) with $k=2$ and $\bar{\gamma}_{out,L}$ is the average output SNR derived in (10).

C. Amount of Fading (AoF) and Spectral Efficiency

AoF is a unified measure of the severity of a fading channel and is typically independent of the average fading power [1, (p. 18)]. It expresses the level of the sensitivity of a wireless system to fading and for the EGC output SNR is defined as

$$AoF = \operatorname{var}\left\langle\gamma_{out,L}\right\rangle\Big/\left(\bar{\gamma}_{out,L}\right)^2. \quad (17)$$

AoF can be evaluated using (6) and (10) and it can be used to study the spectral efficiency of a flat fading channel in the very noise (low power) region. In such regions the minimum energy per bit per noise level, required for reliable communication is [14] $(E_b/N_0)_{min} = -1.59175$ dB and the slope of the spectral efficiency curve versus E_b/N_0 in b/s/Hz per 3 dB, at $(E_b/N_0)_{min}$ is given by [14]

$$S_0 = 2/(AoF + 1) \tag{18}$$

III. NUMERICAL RESULTS AND SIMULATION

In this section we provide several representative numerical curves illustrating the EGC output performance over correlated Nakagami-*m* and Rayleigh fading channels, using the analytical results derived in the preceding sections. In order to check the accuracy of the derived formulae, simulations were performed and the results were compared with the corresponding ones from the mathematical analysis.

Fig.1 plots the first branch normalized average output SNR for a dual EGC receiver over correlated Nakagami-*m* channels, versus the correlation coefficient, ρ, for equal ($\bar{\gamma}_1 = \bar{\gamma}_2$), unequal ($\bar{\gamma}_1 = \bar{\gamma}_2/2$) input SNRs and several values of the fading severity *m*. From this figure, it can be observed that the diversity SNR gain increases as ρ increases. Furthermore, and as expected, the receiver performs better with an increase of the *m* parameter, while its performance deteriorates with an unbalance of the input SNRs. Fig. 2 plots the spectral efficiency versus ρ, for several values of *m* and E_b/N_0. In Fig.3, assuming that the correlation is constant among the EGC branches ($\boldsymbol{\Sigma}_{i,j} \equiv \rho$) and that the receiver operates with an exponentially decaying power delay profile $\left(\Omega_i = \Omega_1 e^{-\delta(i-1)}, i = 1,2,\ldots,L\right)$, the first branch normalized average output SNR is plotted as a function of the number of branches for various values of the Nakagami-*m* parameter, the correlation coefficient and the power decay factor δ. Note, that

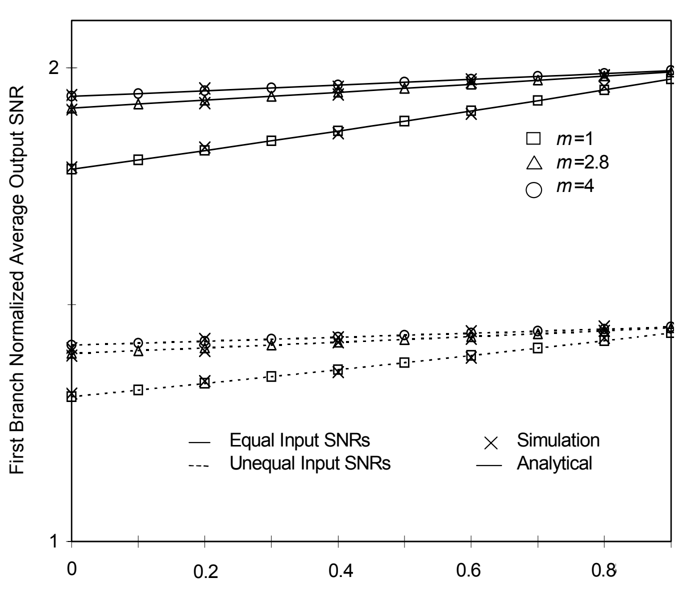

Fig.1: First branch normalized average output SNR for dual EGC versus ρ, for $\bar{\gamma}_1 = \bar{\gamma}_2$, $\bar{\gamma}_1 = \bar{\gamma}_2/2$

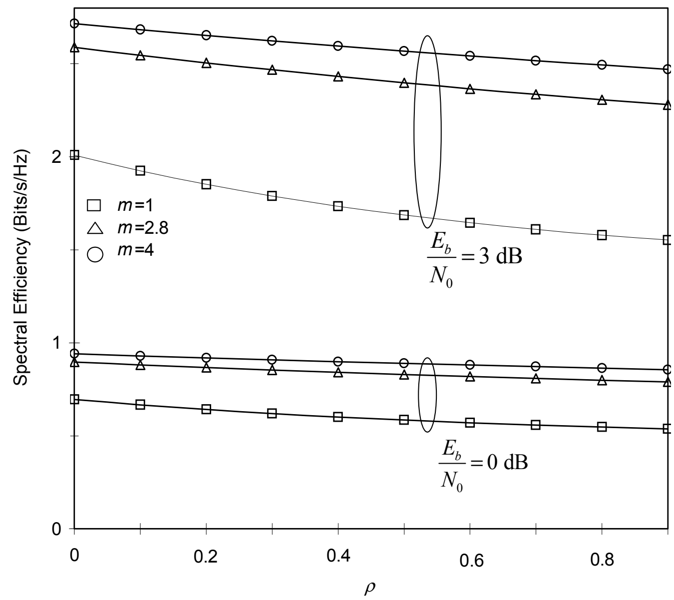

Fig.2: Spectral efficiency of dual EGC versus ρ.

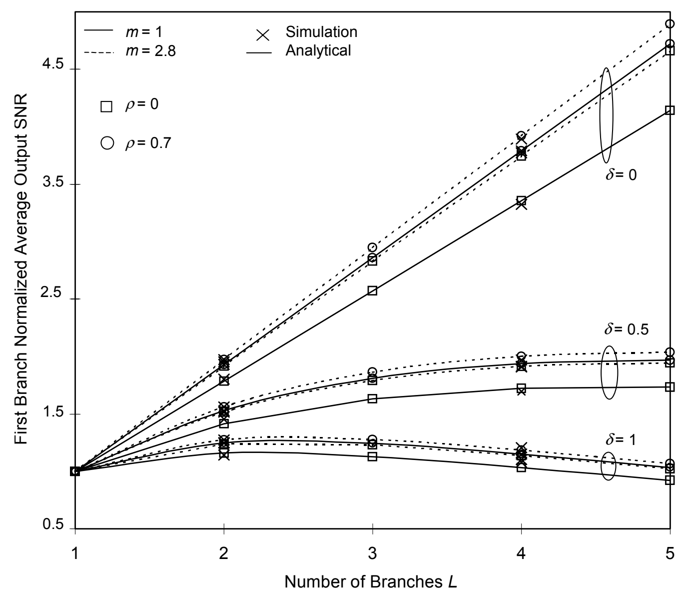

Fig.3: First branch normalized average output SNR with constant correlation and exponentially power delay profile, versus the number of branches *L*.

the combining loss of the receiver gets more accentuated as δ increases.

IV. CONCLUSIONS

In this paper, properties of the predetection EGC output SNR over correlated Nakagami-*m* fading channels are presented. Closed-form expressions for important statistical parameters, such as the *k*-moment and the *k*-central moment are derived. Performance parameters, such as the AoF and the spectral efficiency of the combiner in the high noise region are also investigated. Extensive computer simulations validated the accuracy of the proposed mathematical analysis and numerical results depict clearly the effect of the fading correlation, the fading severity and input SNRs unbalancing on the EGC output performance. The methodology presented in this paper is considered to be a useful and accurate tool for the unified performance analysis of wireless communications systems, operated over correlated generalized-fading channels, and employing EGC.

REFERENCES

[1] M. K. Simon and M.-S. Alouini, *Digital Communication over Fading Channels.* 1st ed., New York: Wiley, 2000.

[2] G. M. Vitetta, U. Mengali, and D. P. Taylor, "An error probability formula for noncoherent orthogonal binary FSK with dual diversity on correlated Rician channels," *IEEE Commun. Lett*., pp. 43-45, Feb. 1999.

[3] L. Fang, G. Bi, and A. C. Kot, "New method of performance analysis for diversity reception with correlated Rayleigh-fading signals," *IEEE Trans. Veh. Technol*., vol. 49, pp. 1807-1812, Sep. 2000.

[4] R. K. Mallik, M. Z. Win, and J. H. Winters, "Performance of dual-diversity EGC in correlated Rayleigh fading with unequal branch SNRs," *IEEE Trans. Commun*., vol. 50, pp. 1041-1044, July 2002.

[5] Y.-K. Ko, M.-S. Alouini, and M. K. Simon, "Average SNR of dual selection combining over correlated Nakagami-*m* fading channels," *IEEE Commun. Lett*., vol. 4, pp. 12-14, Jan. 2000.

[6] D. A. Zogas, G. K. Karagiannidis, and S. A. Kotsopoulos, "On the average output SNR in selection combining with three correlated branches over *m*-fading channels," to be published in in *IEEE Trans. Wirel. Commun*.

[7] M. Nakagami, "The *m*-distribution – A general formula if intensity distribution of rapid fading," in *Statistical Methods in Radio Wave Propagation*, W. G. Hoffman, Ed. Oxford, U.K.: Pergamon, 1960.

[8] C. W. Helstrom, *Probability and Stochastic Processes for Engineers*. 2nd ed. New York: Macmillan, 1991.

[9] M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables,* 9th ed., New York: Dover, 1972.

[10] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 5th ed. New York: Academic, 1994.

[11] G. K. Karagiannidis, D. A. Zogas, and S. A. Kotsopoulos, "An efficient approach to multivariate Nakagami-*m* distribution using Green's matrix approximation," *IEEE Trans. Wirel. Commun.*, vol. 2, Sep. 2003

[12] G. K. Karagiannidis, D. A. Zogas, and S. A. Kotsopoulos, "On the multivariate Nakagami-*m* distribution with exponential correlation," *IEEE Trans. Commun*., vol. 51, pp. 1240-1244, Aug. 2003.

[13] R. Nabben, "On Green's matrices of trees," *SIAM J. Matrix Anal. Appl*., vol. 4, pp. 1014–1026, 2000.

[14] S. Shamai and S. Verdu, " The impact of frequency-flat fading on the spectral efficiency of CDMA," *IEEE Trans. Inform. Theory*., vol. 47, pp. 1302-1327, May 2001.