

Lower and Upper Bounds for the Generalized Marcum and Nuttall Q -Functions

Sotirios K. Mihos, Vasilios M. Kapinas and George K. Karagiannidis

Department of Electrical and Computer Engineering

Aristotle University of Thessaloniki

Thessaloniki, Greece

Email: sotmihos@ee.auth.gr, {kapinas, geokarag}@auth.gr

Abstract—Monotonicity criteria are established for the generalized Marcum Q -function $Q_M(\alpha, \beta)$ and the standard Nuttall Q -function $Q_{M,N}(\alpha, \beta)$. Specifically, we present that $Q_M(\alpha, \beta)$ is monotonically increasing with regard to its order M , for all ranges of the parameters α, β , whereas $Q_{M,N}(\alpha, \beta)$ possesses analogous monotonicity behavior with respect to $M + N$, under the assumptions of $\alpha \geq 1$ and constant difference $M - N \geq 1$. For the normalized Nuttall Q -function $Q_{M,N}(\alpha, \beta)$, we also state the same monotonicity criterion without the necessity of restricting the range of the parameter α . By exploiting these results, we propose closed-form upper and lower bounds for the standard and normalized Nuttall Q -functions, which for the latter case seem to be very tight. Furthermore, concerning the generalized Marcum Q -function, specific tight upper and lower bounds, that have already been proposed in the literature for the case of integer M , are appropriately utilized in order to extend its validity over real values of M . The offered theoretical results can be efficiently applied in the study of digital communications over fading channels, the capacity analysis of multiple-input multiple-output (MIMO) channels and the decoding of turbo or low-density parity-check (LDPC) codes.

I. INTRODUCTION

A. The Generalized Marcum Q -Function

The generalized Marcum Q -function or detection probability has a long history in radar communications and particularly in the study of target detection by pulsed radars with single or multiple observations [1]. Additionally, it is strongly associated with the evaluation of the false alarm probabilities and the error performance analysis in digital communication problems dealing with coherent, differentially coherent and noncoherent detection over fading channels [2]-[4].

Aside from these applications, the generalized Marcum Q -function presents a variety of interesting statistical interpretations. Most indicatively, for integer M , it is the complementary cumulative distribution function (CCDF) of a noncentral chi-square random variable with $2M$ degrees of freedom [5]. Similar relations involving the bivariate Rayleigh, generalized Rayleigh, Rician and Poisson cumulative distribution functions (CDF's), can be found in the literature. Finally, in a recent work [6], a new association has been derived between the generalized Marcum Q -function and a probabilistic comparison of two independent Poisson random variables. The generalized Marcum Q -function of real order $M > 0$, is defined by the

integral [7]

$$Q_M(\alpha, \beta) \triangleq \frac{1}{\alpha^{M-1}} \int_{\beta}^{\infty} x^M e^{-\frac{x^2+\alpha^2}{2}} I_{M-1}(\alpha x) dx, \quad (1)$$

where $I_{M-1}(\cdot)$ is the $(M-1)$ th order modified Bessel function of the first kind [8, Eq. (9.6.3)] and α, β are non-negative real parameters. For $M = 1$, it reduces to the popular standard (or first-order) Marcum Q -function $Q_1(\alpha, \beta)$.

More than thirty algorithms have been proposed in the literature for the numerical calculation of the generalized Marcum Q -function, either with infinite series expansions and recursive relations or with approximations, among them [6], [9]-[11]. However, recursive formulas very often require a large number of iterations to yield sufficient accuracy for very small error probabilities, whereas asymptotic expressions may give values that lie above or below the exact value of the function. In [12], the generalized Marcum Q -function of integer order M has been expressed as a single integral with finite limits which is computationally more desirable relatively to other methods suggested previously. Nevertheless, the integral cannot be solved analytically and hence particular numerical integration techniques have to be applied, introducing thereby an approximation error in its computation. In [5], an exact representation for $Q_M(\alpha, \beta)$, when M is an odd multiple of 0.5, was given as a finite sum of tabulated functions, assuming that $\beta^2 > \alpha^2 + 2M$. This result was recently enhanced in [13] to a single expression that remains accurate over all ranges of the parameters α, β .

Close inspection of the issues mentioned above, render the existence of upper and lower bounds a matter of essential importance in the calculation of (1). Several types of bounds for the generalized Marcum Q -function have been suggested so far [13]-[16]. However, all the previously mentioned works consider only integer values of M , whereas in many applications this requirement doesn't hold. According to [17, Ch. 4.4.2], it would be desirable to obtain alternative representations for $Q_M(\alpha, \beta)$, regardless of whether M is integer or not. Furthermore, a statistical interpretation of $Q_{M-\mu}(\alpha, \beta)$ for $M \in \mathbb{N}^1$ and $\mu = 0.5$ is given in [18], where it is related to main probabilistic characteristics of $2(M - \mu)$

¹Throughout the manuscript, we use \mathbb{N} for the representation of the positive integers set.

random variables. Finally, in [19]-[20], the order M of the generalized Marcum Q -function, involved in the energy detection in various radiometer and cognitive radio applications, is expressed as the product of the integration time and the receiver bandwidth, thus implying that, in general, M is a noninteger quantity.

In this paper, we extend the validity of [13, Eq. (16)]

$$\mathcal{Q}_{M-0.5}(\alpha, \beta) < \mathcal{Q}_M(\alpha, \beta) < \mathcal{Q}_{M+0.5}(\alpha, \beta), \quad M \in \mathbb{N}, \quad (2)$$

by providing a monotonicity formalization for the M th order generalized Marcum Q -function, considering that M is a positive real. Obviously, this enables one to utilize the exact values of $\mathcal{Q}_M(\alpha, \beta)$ derived in [13], in order to obtain tight upper and lower bounds for $\mathcal{Q}_M(\alpha, \beta)$, for the case where M is not necessarily integer.

B. The Nuttall Q -Function

Another function that arises in many applications of digital communication problems is the Nuttall Q -function, defined by the integral [21]

$$\mathcal{Q}_{M,N}(\alpha, \beta) \triangleq \int_b^\infty x^M e^{-\frac{x^2+\alpha^2}{2}} I_N(\alpha x) dx, \quad (3)$$

which, for the special case of $M = N + 1$, is related to the generalized Marcum Q -function of $(N + 1)$ th order according to [17]

$$\mathcal{Q}_{N+1,N}(\alpha, \beta) = \alpha^N \mathcal{Q}_{N+1}(\alpha, \beta). \quad (4)$$

From (4), it follows that $\mathcal{Q}_{1,0}(\alpha, \beta)$ falls into the standard Marcum Q -function $\mathcal{Q}_1(\alpha, \beta)$.

Typical applications involving the Nuttall Q -function, that lay within the telecommunications field, include the error probability performance of noncoherent digital communication over Nakagami fading channels with interference, the outage probability of wireless communication systems with a minimum signal power constraint, the capacity analysis of multiple-input multiple-output (MIMO) Rician channels and finally the extraction of the required log-likelihood ratio for the decoding of turbo or low-density parity-check codes (LDPC) [22]-[24].

To the best of the authors' knowledge, no bounds for the Nuttall Q -function exist in the literature so far. In the rest of this paper, we firstly derive a closed-form expression for the calculation of $\mathcal{Q}_{M,N}(\alpha, \beta)$ where M, N are odd multiples of 0.5 with $M \geq N$. Then, we proceed with the establishment of an appropriate monotonicity criterion that reveals the relation between the Nuttall Q -function and the sum $M + N$. Finally, we go through the utilization of these results in order to derive lower and upper bounds for $\mathcal{Q}_{M,N}(\alpha, \beta)$, for the case where $M \geq N + 1$.

II. MONOTONICITY OF THE GENERALIZED MARCUM Q -FUNCTION

Recently, the authors in [13, Eq. (11)], following a geometric approach, presented a novel closed-form formula for

the evaluation of $\mathcal{Q}_M(\alpha, \beta)$, for the case where M is an odd multiple of 0.5 and $\alpha > 0, \beta \geq 0$, as

$$\begin{aligned} \mathcal{Q}_M(\alpha, \beta) &= \frac{1}{2} \operatorname{erfc} \left(\frac{\beta + \alpha}{\sqrt{2}} \right) + \frac{1}{2} \operatorname{erfc} \left(\frac{\beta - \alpha}{\sqrt{2}} \right) \\ &+ \frac{1}{\alpha \sqrt{2\pi}} \sum_{k=0}^{M-1.5} \frac{\beta^{2k}}{2^k} \sum_{q=0}^k \frac{(-1)^q (2q)!}{(k-q)! q!} \\ &\cdot \left\{ \sum_{i=0}^{2q} \frac{1}{(\alpha\beta)^{2q-i} i!} \left[(-1)^i e^{-\frac{(\beta-\alpha)^2}{2}} \right. \right. \\ &\left. \left. - e^{-\frac{(\beta+\alpha)^2}{2}} \right] \right\}, \end{aligned} \quad (5)$$

where $\operatorname{erfc}(\cdot)$ is the complementary error function [8, Eq. (7.1.2)]. This representation involves only elementary functions and is convenient for evaluation both numerically and analytically. For the trivial case of $\alpha = 0$, exact values of the generalized Marcum Q -function can be obtained from [13, Eq. (12)]

$$\begin{aligned} \mathcal{Q}_M(0, \beta) &= \operatorname{erfc} \left(\frac{\beta}{\sqrt{2}} \right) + \frac{e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}} \\ &\cdot \sum_{k=0}^{M-1.5} \frac{\beta^{2k+1}}{2^{k-1}} \sum_{q=0}^k \frac{(-1)^q}{(k-q)! q! (2q+1)}. \end{aligned} \quad (6)$$

Results (5), (6) along with (2), define tight upper and lower bounds for the generalized Marcum Q -function of integer order. It seems apparent, that in order to derive bounds for $\mathcal{Q}_M(\alpha, \beta)$ of real order M , a stricter inequality, involving all values of M , has to be established. Such a generalization concept can be formalized through the following theorem

Theorem 1: (Monotonicity of the generalized Marcum Q -function) The generalized Marcum Q -function $\mathcal{Q}_M(\alpha, \beta)$ is a monotonically strictly increasing function with respect to its order M .

Proof: The proof is provided in [25]. ■

Theorem 1 implies the following inequalities, serving as bounds for the generalized Marcum Q -function $\mathcal{Q}_M(\alpha, \beta)$ of real order M

$$\begin{aligned} \mathcal{Q}_{\lfloor M \rfloor - 0.5}(\alpha, \beta) &< \mathcal{Q}_M(\alpha, \beta) < \mathcal{Q}_{\lfloor M \rfloor + 0.5}(\alpha, \beta), \\ &\text{for } 0 \leq \delta_M < 0.5 \\ \mathcal{Q}_{\lceil M \rceil - 0.5}(\alpha, \beta) &< \mathcal{Q}_M(\alpha, \beta) < \mathcal{Q}_{\lceil M \rceil + 0.5}(\alpha, \beta), \\ &\text{for } 0.5 < \delta_M < 1, \end{aligned} \quad (7)$$

where $\lfloor M \rfloor$ denotes the largest integer less than or equal to M , $\lceil M \rceil$ is the least integer greater than M and $\delta_M = M - \lfloor M \rfloor$, with $\delta_M \in [0, 1)$. For $\delta_M = 0$ one can use directly the representations (5), (6) for the evaluation of $\mathcal{Q}_M(\alpha, \beta)$.

Concerning their tightness, these bounds have been demonstrated in [13] to outperform other existing ones, for the case of integer M . Therefore, for the case of real M , one can expect even further enhancement in the strictness of either the lower bound (for $\delta_M > 0.5$) or the upper one (for $\delta_M < 0.5$).

In Fig. 1, the generalized Marcum Q -function has been plotted versus its order M , for indicative values of the parameters α, β .

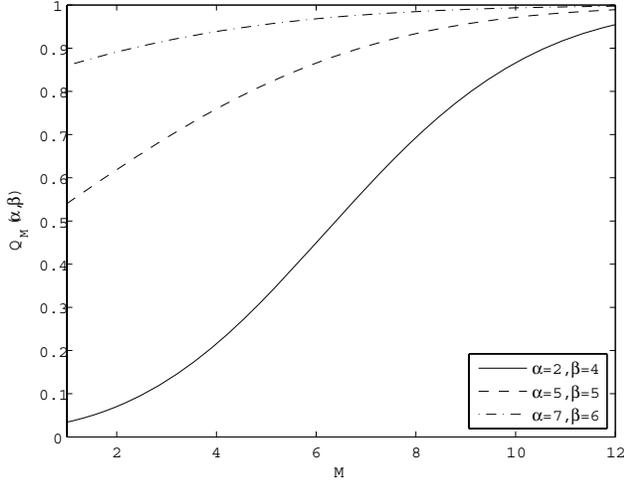


Fig. 1. Monotonicity of the generalized Marcum Q -function $Q_M(\alpha, \beta)$ versus M , for the cases of $\alpha < \beta$, $\alpha = \beta$ and $\alpha > \beta$.

III. MONOTONICITY OF THE NUTTALL Q -FUNCTION

A. A Novel Closed-Form Representation for $Q_{M,N}(\alpha, \beta)$

So far, closed-form expressions for the Nuttall Q -function are not available in the literature. In this section, we derive such a representation, for the case where M, N are odd multiples of 0.5 and $M \geq N$. This last result will be particularly useful in the derivation of bounds for $Q_M(\alpha, \beta)$, for integers M, N , as it will be presented later.

The modified Bessel function of the first kind in the integral of (3), can be expressed by the finite sum [26, Eq. (8.467)]

$$I_N(z) = \frac{(-1)^n (2z)^{-n+\frac{1}{2}}}{\sqrt{\pi} e^z} \sum_{k=0}^{n-1} \frac{(n-k)_{n-1} (2z)^k}{k!} \cdot [1 - (-1)^k e^{2z}], \quad n = N + 0.5 \in \mathbb{N}, \quad (8)$$

with $(\cdot)_\mu$ denoting the Pochhammer's symbol [8, Eq. (6.1.22)]. After substituting (8) in the integral of (3), we obtain

$$Q_{M,N}(\alpha, \beta) = \frac{(-1)^n (2\alpha)^{-n+\frac{1}{2}}}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{(n-k)_{n-1} (2\alpha)^k}{k!} \cdot \mathcal{I}_{m,n}^k(\alpha, \beta), \quad n = N + 0.5 \in \mathbb{N}, \quad m = M + 0.5 \in \mathbb{N}, \quad (9)$$

where the term $\mathcal{I}_{m,n}^k(\alpha, \beta)$ can be given from the expression (the proof is provided in [25])

$$\begin{aligned} \mathcal{I}_{m,n}^k(\alpha, \beta) &= (-1)^{k+1} \sum_{l=0}^{m-n+k} \binom{m-n+k}{l} 2^{\frac{l-1}{2}} \alpha^{m-n+k-l} \\ &\cdot \left[(-1)^{m-n-l-1} \Gamma\left(\frac{l+1}{2}, \frac{(\beta+\alpha)^2}{2}\right) \right. \\ &+ [\text{sgn}(\beta-\alpha)]^{l+1} \Gamma\left(\frac{l+1}{2}, \frac{(\beta-\alpha)^2}{2}\right) \\ &+ \left. (1 - [\text{sgn}(\beta-\alpha)]^{l+1}) \Gamma\left(\frac{l+1}{2}\right) \right]. \quad (10) \end{aligned}$$

In (10), $\binom{\lambda}{\mu}$ is the generalized binomial coefficient [27, app. I.5], $\Gamma(\cdot, \cdot)$ is the upper incomplete Gamma function [8, Eq. (6.5.3)] and $\text{sgn}(\cdot)$ is the signum function defined as $\text{sgn}(x) = x/|x|, \forall x \neq 0; = 0, \text{ for } x = 0$. Therefore, the Nuttall Q -function $Q_{M,N}(\alpha, \beta)$, for half-odd integers M, N , can be evaluated by the closed-form expressions (9) and (10).

B. Upper and Lower Bounds for the Nuttall Q -Function

To the best of the authors' knowledge, neither upper nor lower bounds exist in the literature for the Nuttall Q -function. In this section, we propose bounds of both types for $Q_{M,N}(\alpha, \beta)$, for the case where $\alpha \geq 1$ and $M \geq N + 1$, by using (9) and the following theorem

Theorem 2: (Monotonicity of the Nuttall Q -function) The Nuttall Q -function $Q_{M,N}(\alpha, \beta)$ is a monotonically strictly increasing function with respect to the sum $M + N$, under the requirements of $\alpha \geq 1$ and constant difference $M - N \geq 1$.

Proof: The proof is provided in [25]. ■

Theorem 2 enables us to write the inequalities

$$\begin{aligned} Q_{M,N}(\alpha, \beta) &> Q_{M-0.5, N-0.5}(\alpha, \beta) \\ Q_{M,N}(\alpha, \beta) &< Q_{M+0.5, N+0.5}(\alpha, \beta), \quad (11) \end{aligned}$$

where $\alpha \geq 1$. Obviously, assuming that the indices of $Q_{M,N}$ satisfy the inequality $M \geq N + 1$, then for constant difference $M - N$, this requirement is met as well by $Q_{M-0.5, N-0.5}(\alpha, \beta)$ and $Q_{M+0.5, N+0.5}(\alpha, \beta)$.

It is important to mention here, that if we define the *normalized Nuttall Q -function* with respect to its parameter α as

$$Q_{M,N}(\alpha, \beta) \triangleq \frac{Q_{M,N}(\alpha, \beta)}{\alpha^N}, \quad (12)$$

then its monotonicity property can be determined by the next theorem

Theorem 3: (Monotonicity of the normalized Nuttall Q -function) The normalized Nuttall Q -function $Q_{M,N}(\alpha, \beta)$ is a monotonically strictly increasing function with respect to the sum $M + N$, under the requirement of constant difference $M - N \geq 1$.

Proof: The proof is provided in [25]. ■

Apparently, Theorem 3 implies that similar inequalities to (11) hold for $Q_{M,N}$ as well, without the necessity of restricting the range of a

$$\begin{aligned} Q_{M,N}(\alpha, \beta) &> \frac{Q_{M-0.5, N-0.5}(\alpha, \beta)}{\alpha^{N-0.5}} \\ Q_{M,N}(\alpha, \beta) &< \frac{Q_{M+0.5, N+0.5}(\alpha, \beta)}{\alpha^{N+0.5}}. \quad (13) \end{aligned}$$

In Figs. (2), (3) the normalized Nuttall Q -function along with its lower and upper bounds are depicted versus β for different values of α and M , respectively, considering $N = 3$ for both graphs. It is evident, that the bounds proposed in (13) are very tight.

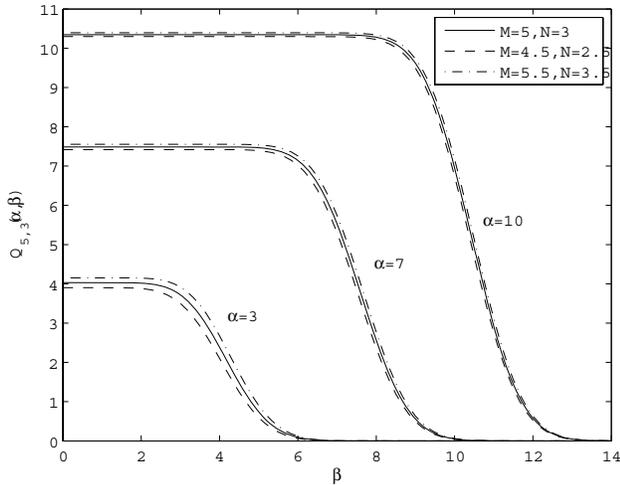


Fig. 2. Monotonicity of the normalized Nuttall Q -function $Q_{5,3}(\alpha, \beta)$ versus β , for three cases of α .

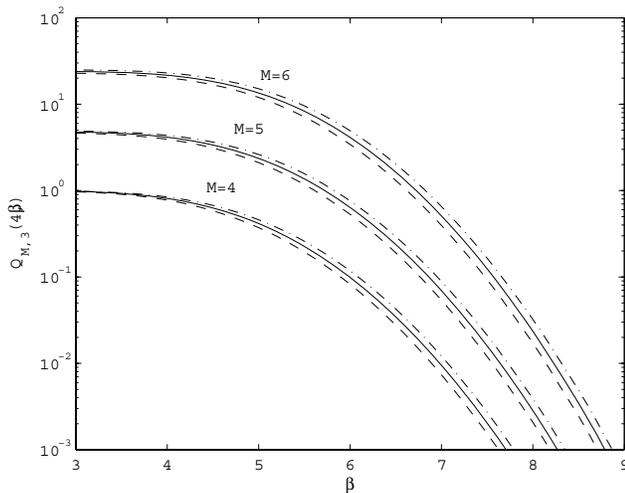


Fig. 3. Monotonicity of the normalized Nuttall Q -function $Q_{M,3}(4, \beta)$ versus β for three cases of M .

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REFERENCES

- [1] J. I. Marcum, "A statistical theory of target detection by pulsed radar," *IRE Trans. Inform. Theory*, vol. IT-6, pp. 59–267, Apr. 1960.
- [2] J. Proakis, *Digital Communications*, 3rd ed. New York: McGraw-Hill, 1995.
- [3] M. K. Simon and M.-S. Alouini, "A unified approach to the probability of error for noncoherent and differentially coherent modulations over generalized fading channels," *IEEE Trans. Commun.*, vol. 46, pp. 1625–1638, Dec. 1998.

- [4] F. F. Digham, M.-S. Alouini, and M. K. Simon, "On the energy detection of unknown signals over fading channels," in *Proc. IEEE Int. Conf. on Commun. (ICC'03)*, vol. 5, pp. 3575–3579, Alaska, USA, May 2003.
- [5] A. H. M. Ross, "Algorithm for calculating the noncentral chi-square distribution," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1327–1333, May 1999.
- [6] G. V. Weinberg, "Poisson Representation and Monte Carlo Estimation of Generalized Marcum Q -Function," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 42, pp. 1520–1531, Oct. 2006.
- [7] J. I. Marcum, *Table of Q Functions*, U.S. Air Force Project RAND Res. Memo. M-339, ASTIA Document AD 1165451, Rand Corp., Santa Monica, CA, Jan. 1950.
- [8] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th ed. New York, NY: Dover Press, 1972.
- [9] G. M. Dillard, "Recursive computation of the generalized Q -function," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-9, pp. 614–615, July 1973.
- [10] S. Parl, "A new method of calculating the generalized Q function," *IEEE Trans. Inform. Theory*, vol. IT-26, pp. 121–124, Jan. 1980.
- [11] D. A. Shnidman, "The calculation of the probability of detection and the generalized Marcum Q -function," *IEEE Trans. Inform. Theory*, vol. 35, pp. 389–400, Mar. 1989.
- [12] M. K. Simon, "A new twist on the Marcum Q -function and its application," *IEEE Commun. Lett.*, vol. 2, pp. 39–41, Feb. 1998.
- [13] R. Li and P. Y. Kam, "Computing and Bounding the Generalized Marcum Q -Function via a Geometric Approach," in *Proc. IEEE Int. Symp. Inform. Theory (ISIT'06)*, Seattle, USA, pp. 1090–1094, July 2006.
- [14] S. S. Rappaport, "Computing approximations for the generalized Q function and its complement," *IEEE Trans. Inform. Theory*, vol. IT-17, pp. 497–498, July 1971.
- [15] M. K. Simon and M.-S. Alouini, "Exponential-type bounds on the generalized Marcum Q -function with application to error probability analysis over fading channels," *IEEE Trans. Commun.*, vol. 48, pp. 359–366, Mar. 2000.
- [16] A. Annamalai and C. Tellambura, "Cauchy-Schwartz bound on the generalized Marcum Q -function with applications," *J. Wireless Commun. Mobile Comput.*, vol. 1, pp. 243–253, Mar. 2001.
- [17] M. K. Simon and M.-S. Alouini, *Digital Communication Over Fading Channels*, 2nd ed. Hoboken, NJ: Wiley- Interscience, 2005.
- [18] G. Di. Blasio and A. Neri, "Modified Q -Functions and Their Use in Detection Analysis," *IEEE Trans. Inform. Theory*, vol. IT-37, no. 4, pp. 1123–1142, July 1991.
- [19] C. W. Helstrom, "Approximate inversion of Marcum's Q -function," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 34, pp. 317–319, Jan. 1998.
- [20] A. Pandharipande and J.-P. M. G. Linnartz, "Performance Analysis of Primary User Detection in a Multiple Antenna Cognitive Radio," in *Proc. IEEE Int. Conf. on Commun. (ICC'07)*, pp. 6482–6486, Glasgow, Scotland, June. 2007.
- [21] A. Nuttall, *Some Integrals Involving the Q-function*, Technical Report 4297, Naval Underwater Systems Center, New London, CT, April 1972.
- [22] H. C. Yang and M.-S. Alouini, "Closed Form Formulas for the Outage Probability of Wireless Communication Systems with a Minimum Signal Power Constraint," *IEEE Trans. Vehic. Technol.*, vol. 51, pp. 1689–1698, Nov. 2002.
- [23] A. Maaref and S. Aissa, "Joint and Marginal Eigenvalue Distributions of (Non) Central Wishart Matrices and PDF-based Approach for Characterizing the Capacity Statistics of MIMO Ricean and Rayleigh Fading Channels," *IEEE Trans. Wirel. Commun.*, vol. 6, pp. 3607–3619, Oct. 2007.
- [24] K.-P. Ho and H. C. Wang, "Soft-Decoding of Low-Density Parity Check Codes for DPSK Signals," in *Proc. Opt. Fiber Commun. Conf. 2007. Technical Digest (OFC/NFOEC'07)*, pp. 1–3, California, USA, Mar. 2007.
- [25] S. K. Mihos, V. M. Kapinas, and G. K. Karagiannidis, "On the Monotonicity of the Generalized Marcum and Nuttall Q -Functions," *IEEE Trans. Inform. Theory*, submitted.
- [26] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. San Diego, CA: Academic Press, 2000.
- [27] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series. Elementary Functions*, vol. 1, 3rd ed. New York, NY: Gordon and Breach, 1992.