On the Distribution of the Sum of Gamma-Gamma Variates and Application in MIMO Optical Wireless Systems

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***Abstract*— We present a novel approach for the accurate approximation, via closed-form expressions, of the distribution of the sum of independent and not necessarily identically distributed Gamma-Gamma (GG) variates. It is shown that the probability density function (PDF) of the GG sum can be efficiently approximated either by the PDF of a single GG distribution, or by a finite weighted sum of PDFs of GG distributions. Ascertaining on this result, the performance of multiple input multiple output (MIMO) optical wireless (OW) systems is investigated and approximative closed-form expressions for important performance metrics are derived. Numerical results and simulations illustrate the accuracy of the proposed approach.**

I. Introduction

A distribution which has recently attracted the interest within the research community due to its involvement in various communication systems, is the so-called Gamma-Gamma (GG) distribution. This distribution is produced from the product of two independent Gamma random variables (RVs) and has been introduced as a generalization of the well-known K-distribution [1] (therefore it is also known as Generalized-K). In the past, it has been widely used in a variety of applications, including modeling of various types of land and sea radar clutters [2], as well as to model the effects of the combined fading and shadowing phenomena, encountered in the mobile communications channels [3]. Of particular interest is the application of the GG distribution in optical wireless (OW) systems, where transmission of optical signals through the atmosphere is involved. In such systems, a major performance limiting factor is *turbulence induced fading*, i.e. rapid fluctuations of the irradiance of the propagated optical signals caused by atmospheric turbulence, which can be accurately modeled using the statistics of GG distribution [4].

Although GG channel model is analytically tractable in the performance analysis of various single input single output (SISO) wireless communication systems [3]- [6], difficulties arise when studying the performance of certain diversity schemes in multiple input multiple output (MIMO) wireless systems. Specifically, these difficulties appear when the distribution of the sum of GG variates is required and have their origin in the fact that a straight derivation of this distribution is analytically infeasible due to the involvement of the modified Bessel function of the second kind.

In the past, a limited number of published works dealing with the distribution of the sum of independent GG variates and its application in communications systems, appeared in the literature. In the context of mobile communications, where GG distribution models the statistics of the signal-to-noise ratio (SNR) in the presence of composite multipath/shadowing fading, the maximal ratio combining (MRC) receiver has been investigated in [7]. However, the expressions that were derived in this important work, for the statistics of the SNR at the output of the combiner, do not hold[1]. In the context of OW communications, the statistics of the sum of GG variates have been used in the performance analysis of MIMO systems operating over GG turbulence model and employing equal gain combining (EGC) at the receiver. In [8], the performance of such a system is investigated over identical OW links, using an infinite power series respresentation for the probability density function (PDF) of the turbulence-induced fading term at the output of the receiver. Although this approach allows the derivation of simple and accurate expressions at the high SNR regime, it is not computationally attractive when the number of the transmit/receive apertures increases and/or the underlying OW links are non identically distributed.

In this paper, we address this cumbersome statistical problem by applying a novel and simpler approach. Ascertaining on, a) GG distribution is derived from the product of two independently distributed Gamma RVs, and b) the distribution of the sum of Gamma variates is analytically tractable, we present novel closed-form expressions that approximate efficiently the PDF of the distribution of the sum of GG variates. Our analysis encompasses the case where the variates involved in the sum are identically distributed, as well as the case of non identical variates. Furthermore, in order to reveal the importance of the proposed statistical formulation, we study the performance of MIMO OW systems operating over strong turbulence channels and employing EGC at the receiver. For such systems, closed-form expressions that approximate significant system performance metrics, such as bit error rate (BER) and outage probability, are obtained.

The remainder of the paper is organized as follows. After

[1]In [7, Eq. A-2], it is assumed that the parameter $a = k - m$ is not an integer. However this comes in contradiction with the assumption that the parameters k and m in [7, Eq. A-4] are integers, which results to the incorrect expressions of [7, Eq. 11] and [7, Eq. 12].

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a brief introduction of the GG distribution in Section II, novel closed-form expressions that approximate the PDF of the sum of GG variates are obtained in Section III. Furthermore, in Section IV, the obtained results are applied to derive approximative closed-form expressions for the performance evaluation of MIMO OW systems operating over strong turbulence channels. Finally, in Section V, useful concluding remarks are provided.

II. The GG Distribution

Let $\gamma \geqslant 0$, then the PDF of a three-parameter GG RV is given by [3]

$$f_{\gamma}\left(\gamma;k,m,\Omega\right)=\frac{2(km)^{\frac{k+m}{2}}\gamma^{\frac{k+m}{2}-1}}{\Gamma\left(m\right)\Gamma\left(k\right)\Omega^{\frac{k+m}{2}}}K_{k-m}\left[2\left(\frac{km}{\Omega}\gamma\right)^{1/2}\right] \tag{1}$$

where $k \geqslant 0$ and $m \geqslant 0$ are the distribution shaping parameters, $K_{\nu}\left(\cdot\right)$ is the modified Bessel function of order ν [9, 8.407/1], $\Gamma\left(\cdot\right)$ is the Gamma function [9, 8.310/1] and Ω is related with the mean as $\mathbf{E}\left[\gamma\right]=\Omega$, with $\mathbf{E}\left[\cdot\right]$ denoting expectation. The n-th moment of γ is given as [3]

$$\mathbf{E}\left[\gamma^{n}\right]=\xi^{-n}\frac{\Gamma\left(k+n\right)\Gamma\left(m+n\right)}{\Gamma\left(k\right)\Gamma\left(m\right)}, \tag{2}$$

where $\xi=\frac{km}{\Omega}$. Its cumulative density function (CDF) has been expressed using the well-known Meijer's functions [9, Eq. 9.301]. According to [5, Eq. 7] and using [9, Eq. 9.31/5], the CDF is derived as

$$F_{\gamma}\left(\gamma;k,m,\Omega\right)=\frac{1}{\Gamma\left(k\right)\Gamma\left(m\right)}G_{1,3}^{2,1}\left[\xi\gamma\left|\begin{array}{c}1\\k,m,0\end{array}\right.\right] \tag{3}$$

where $G\left[\cdot\right]$ is Meijer's G function.

It is important to note that the GG distribution can be derived from the product of two independent RVs, x and y as [10]

$$\gamma=xy, \tag{4}$$

which are both Gamma distributed with PDF given by

$$f\left(i;m_{i},\eta_{i}\right)=\frac{i^{m_{i}-1}}{\eta_{i}^{m_{i}}\Gamma\left(m_{i}\right)}\exp\left(-\frac{i}{\eta_{i}}\right),\ i=x,y \tag{5}$$

and set of parameters $(m_x=k,\eta_x=1/k)$ and $(m_y=m,\eta_y=\Omega/m)$ respectively[2].

III. Efficient Approximation to the Sum of GG variates

Let us consider L independent GG variates denoted by $\{\gamma_l\}_{l=1}^{L}$, each having shaping parameters k_l and m_l, and mean Ω_l. The sum of L GG variates, S_γ, is defined as

$$S_{\gamma}\triangleq\sum_{l=1}^{L}\gamma_{l} \tag{6}$$

[2]It follows from symmetry that it is equivalent to consider the set of parameters $\left(m_x=k,\eta_x=\frac{\Omega}{k}\right)$ and $\left(m_y=m,\eta_y=\frac{1}{m}\right)$.

or equivalently, by using (4), as

$$S_{\gamma}=\sum_{l=1}^{L}x_{l}y_{l} \tag{7}$$

where x_l and y_l are Gamma RVs with parameters $(k_l,1/k_l)$ and $(m_l,\Omega_l/m_l)$ respectively. Eq. (7) can be rewritten as

$$S_{\gamma}=\frac{\left(\sum\limits_{l=1}^{L}x_{l}\right)\left(\sum\limits_{l=1}^{L}y_{l}\right)}{L}+\frac{1}{L}\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left(x_{i}-x_{j}\right)\left(y_{i}-y_{j}\right) \tag{8}$$

A. Identical Variates

When the variates of the sum in (6) are independent and identically distributed (i.i.d.) (i.e. $k_l=k$, $m_l=m$, $\Omega_l=\Omega$), $\{x_l\}_{l=1}^{L}$ and $\{y_l\}_{l=1}^{L}$ are also identically distributed. Hence, according to (8), the unknown distribution of S_γ can be approximated by the distribution of the RV $\hat{S}_\gamma$, which is defined as

$$S_{\gamma}\approx\hat{S}_{\gamma}=\frac{\left(\sum\limits_{l=1}^{L}x_{l}\right)\left(\sum\limits_{l=1}^{L}y_{l}\right)}{L} \tag{9}$$

with the approximation error, ε, given by

$$\varepsilon=\frac{1}{L}\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left(x_{i}-x_{j}\right)\left(y_{i}-y_{j}\right) \tag{10}$$

Equivalently, (9) can be written as the product of two RVs s_1 and s_2, i.e.

$$\hat{S}_{\gamma}=s_{1}s_{2} \tag{11}$$

where

$$s_{1}=\frac{1}{L}\sum_{l=1}^{L}x_{l} \tag{12}$$

and

$$s_{2}=\sum_{l=1}^{L}y_{l} \tag{13}$$

Since the sum of i.i.d. Gamma variates remains Gamma distributed [11], it is easy to prove that s_1 and s_2 are both Gamma distributed with set of parameters $\left(Lk,\frac{1}{Lk}\right)$ and $\left(Lm,\frac{L\Omega}{Lm}\right)$ respectively. Hence, according to (4), $\hat{S}_\gamma$ will be GG distributed with the shaping parameters

$$k_{\hat{S}_{\gamma}}=Lk \tag{14}$$

$$m_{\hat{S}_{\gamma}}=Lm \tag{15}$$

and mean

$$\Omega_{\hat{S}_{\gamma}}=L\Omega \tag{16}$$

In order to improve the accuracy of the proposed approximation, an adjustment parameter is introduced that modifies its shaping parameters. Specifically, we assume that the maximum

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of the shaping parameters[3], $k_{\hat{S}_\gamma}$, is modified by an adjustment parameter, ε_γ, according to

$$k_{\hat{S}_\gamma} = Lk + \varepsilon_\gamma \tag{17}$$

The adjustment parameter, ε_γ, is evaluated through the following optimization problem

$$\varepsilon_\gamma = \arg\min_{\varepsilon_\gamma} \left| \mathbf{E}\left[\hat{S}_\gamma^\nu\right] - \mathbf{E}\left[S_\gamma^\nu\right] \right|, \nu = 1, ...4 \tag{18}$$

where $\mathbf{E}\left[\hat{S}_\gamma^\nu\right]$ are the moments of the distribution of $\hat{S}_\gamma$, which are derived from (2) using the parameters $\left(k_{\hat{S}_\gamma}, m_{\hat{S}_\gamma}, \Omega_{\hat{S}_\gamma}\right)$, as provided by (17), (15) and (16) respectively; $E\left[S_\gamma^\nu\right]$ are the moments of the distribution of S_γ, calculated using the multinomial expansion, according to

$$\begin{aligned}\mathbf{E}\left[S_\gamma^\nu\right] = \sum_{\nu_1=0}^{\nu} \sum_{\nu_2=0}^{\nu_1} ... \sum_{\nu_{L-1}=0}^{\nu_{L-2}} \binom{\nu}{\nu_1}\binom{\nu_1}{\nu_2}...\binom{\nu_{L-2}}{\nu_{L-1}} \\ \times \mathbf{E}\left[\gamma_1^{\nu-\nu_1}\right]\mathbf{E}\left[\gamma_2^{\nu_1-\nu_2}\right]...\mathbf{E}\left[\gamma_L^{\nu_{L-1}}\right]\end{aligned} \tag{19}$$

and using (2) for the respective parameters (k, m, Ω).

The optimization problem in (18) is a nonlinear multiple function optimization problem, which is difficult to solve analytically, yet not impossible to derive an approximative solution numerically. After applying non-linear regression methods [12], it was found that the adjustment parameter depends on L, k and m with a function of the form of

$$\varepsilon_\gamma (L, k, m) = (L-1)\frac{-0.127 - 0.95k - 0.0058m}{1 + 0.00124k + 0.98m} \tag{20}$$

Hence, using (20) in conjunction with (17), (15) and (16), the parameters of a single GG distribution are defined, which accurately approximates the distribution of the sum of L i.i.d. GG variates.

B. Non-Identical Variates

When the GG variates of the sum in (6) are independent and not identically distributed (i.n.i.d.), but have one shaping parameter in common, (i.e. $k_l = k$, but m_l and Ω_l are different), the unknown PDF of the sum can still be approximated by the PDF of the RV $\hat{S}_\gamma$, as defined by (11).

As in the i.i.d. case, $\hat{S}_\gamma$ can be written as the product of two RVs s_1 and s_2, defined by (12) and (13) respectively. Since $k_l = k$, $\{x_l\}_{l=1}^L$ are i.i.d. and s_1 is Gamma distributed with PDF given by (5) and parameters $\left(Lk, \frac{1}{Lk}\right)$. However, the derivation of the distribution of s_2 is not straightforward, since $\{y_l\}_{l=1}^L$ are not identically distributed. In order to derive the PDF of s_2, the exact closed-form expressions for the sum of non-identical Gamma variates presented in [13] are used. According to this approach, the PDF of s_2 can be written as a nested finite weighted sum of Gamma PDFs,

$$f_{s_2}(z) = \sum_{i=1}^{L}\sum_{j=1}^{m_i} W_L(i,j)\, f_y\left(z; j, \frac{\Omega_i}{m_i}\right) \tag{21}$$

[3]Note that since $K_{-\nu}(x) = K_\nu(x)$, we assume for simplification of the analysis and without loss of generality that $k_{\hat{S}_\gamma} \geqslant m_{\hat{S}_\gamma}$.

where y is a Gamma distributed RV with PDF given by (5) and the weights $W_L(i,j)$ can be evaluated using [13, Eq. (7)] for the set of parameters $\{m_l\}_{l=1}^L$ and $\left\{\frac{\Omega_l}{m_l}\right\}_{l=1}^L$.

The PDF of the product of s_1 and s_2 is evaluated as

$$f_{\hat{S}_\gamma}(z) = \int_0^\infty \frac{1}{x} f_{s_1}\left(x; Lk, \frac{1}{Lk}\right) f_{s_2}\left(\frac{z}{x}\right) dx \tag{22}$$

Using (21) and [9, Eq. 3.471/9], Eq. (22) yields as

$$f_{\hat{S}_\gamma}(z) = \sum_{i=1}^{L}\sum_{j=1}^{m_i} W_L(i,j)\, f_\gamma\left(z; Lk, j, \frac{j\Omega_i}{m_i}\right) \tag{23}$$

where γ is a GG distributed RV. Hence, an efficient approximation to the PDF of the sum of L non-identical GG variates, when one of the shaping parameters remains the same for all variates[4], can be a nested finite weighted sum of GG PDFs.

IV. Application in Optical Wireless Systems

A. System Model

Consider a Multiple Input Multiple Output (MIMO) optical wireless (OW) system where the information signal is transmitted via M apertures and received by N apertures over strong atmospheric turbulence conditions. For the OW system under consideration, it is assumed that the information bits are modulated using On-Off keying (OOK) and transmitted through the M apertures using repetition coding [14]. Moreover, a large field of view is considered for each receiver indicating that multiple transmitters are simultaneously observed by each receiver. This actually leads to the collection of larger amount of background radiation which justifies the use of the AWGN model as a good approximation of the Poisson photon counting detection model [15]. Hence, the received signal at the nth receive aperture is given by

$$r_n = x\eta \sum_{m=1}^{M} I_{mn} + \upsilon_n,\ n = 1, ...N \tag{24}$$

where $x \in \{0,1\}$ represents the information bits, η is the optical-to-electrical conversion coefficient and υ_n is the AWGN with zero mean and variance $\sigma_\upsilon^2 = N_o/2$.

The term I_{mn} denotes the fading coefficient that models the atmospheric turbulence through the optical channel between the mth transmit and the nth receive aperture. Since operation under strong atmospheric turbulence conditions is assumed, according to [4], the parameter which represents the effective number of small scale scatterers can be considered equal to 1. Hence, the optical channel in each transmit-receive pair can be statistically described by a GG distribution with parameters $k = 1$, $m = a_{mn}$ and $\Omega = \mathbf{E}[I_{mn}]$ [4], where a_{mn} is related to the effective number of large scale scatterers. Furthermore, it is assumed that the statistics of the fading coefficients of the underlying channels are statistically independent; an

[4]Note that due to symmetry, the same approximation also holds when $m_i = m$ and k_i are different, by interchanging k_i and m_i in (23), i.e. $m_i = k_i$ and $k = m$.

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assumption which is realistic by placing the transmitter and the receiver apertures just a few centimeters apart [16].

At the receiver side, the received optical signals from the N apertures are combined using equal gain combining (EGC). Hence, the output of the receiver is

$$r = \frac{1}{NM}\sum_{n=1}^{N} r_n = \frac{x\eta}{NM}\sum_{n=1}^{N}\sum_{m=1}^{M} I_{mn} + \upsilon. \qquad (25)$$

Note that a scaling factor of MN appears in (25). The factor M is included in order to ensure that the total transmit power is the same with that of a system with no transmit diversity, while the factor N ensures that the sum of the N receive aperture areas is the same with the aperture area of a system with no receive diversity.

The received electrical SNR of the OW link between the m transmit and n receive aperture, can be defined as [17]

$$h_{mn} = \frac{\eta^2 I_{mn}^2}{N_o}, \qquad (26)$$

while its average as $\mu_{mn} = \frac{\eta^2 E[I_{mn}]^2}{N_o}$. According to the above definitions, the electrical SNR of the combined signal at the output of the receiver, becomes

$$h_T = \frac{\eta^2 (I_T)^2}{N^2 M^2 N_o}, \qquad (27)$$

where $I_T = \sum_{n=1}^{N}\sum_{m=1}^{M} I_{mn}$.

B. Error Analysis

The Bit Error Rate (BER) probability of the MIMO OW system under consideration, assuming perfect Channel State Information (CSI), is given by [15] as

$$P_e = \int_{\boldsymbol{I}} f_{\boldsymbol{I}}(\boldsymbol{I})\, Q\left(\frac{\eta}{2NM\sigma_\upsilon}\sum_{n=1}^{N}\sum_{m=1}^{M} I_{mn}\right) \mathrm{d}\boldsymbol{I} \qquad (28)$$

where $f_{\boldsymbol{I}}(\boldsymbol{I})$ is the joint PDF of the vector $\boldsymbol{I} = (I_{11}, I_{12}, ...I_{MN})$ of length MN. Furthermore, $Q(\cdot)$ is the Gaussian-Q function defined as $Q(y) = \left(1/\sqrt{2\pi}\right)\int_y^\infty \exp\left(-t^2/2\right) dt$ and related to $\mathrm{erfc}(\cdot)$ by $\mathrm{erfc}(\mathrm{x}) = 2\mathrm{Q}\left(\sqrt{2}\mathrm{x}\right)$. Equivalently, Eq. (28) can be evaluated as

$$P_e = \frac{1}{2}\int_0^\infty f_{I_T}(I)\,\mathrm{erfc}\left(\frac{\eta}{2\sqrt{2}\mathrm{NM}\sigma_\upsilon}\mathrm{I}\right)\mathrm{dI} \qquad (29)$$

where $f_{I_T}(I)$ is the PDF of I_T.

1) Independent and Identically Distributed OW Links: When the turbulence induced fading coefficients of the underlying optical links of the MIMO system are independent and identically distributed, i.e. $a_{mn} = a$ and $\mathbf{E}[I_{mn}] = I_o$, the PDF of I_T can be approximated by the PDF of a single GG variate, i.e.

$$f_{I_T}(I) \approx f_\gamma(I; k_T, m_T, \Omega_T) \qquad (30)$$

where $k_T = MNa + \varepsilon_\gamma$, $m_T = MN$, $\Omega_T = MNI_o$ and ε_γ is calculated from (20) for set of parameters equal to $(MN, a, 1)$.

Hence, the BER probability of (29) is approximated by

$$P_e \approx \frac{1}{2}\int_0^\infty f_\gamma(I; k_T, m_T, \Omega_T)\,\mathrm{erfc}\left(\frac{\eta}{2\sqrt{2}\mathrm{NM}\sigma_\upsilon}\mathrm{I}\right)\mathrm{dI} \qquad (31)$$

The integral of (31) can be solved using Meijer's G-functions and their properties. Hence, by substituting the PDF of the GG distribution according to (1), expressing the $K_\nu(\cdot)$ and the $\mathrm{erfc}(\cdot)$ integrands in terms of Meijer's G-function according to [18, Eq. (8.4.23.1)] and [18, Eq. (8.4.14.2)] respectively, and using [18, Eq. (2.24.1.1)], the closed-form solution of (32) yields, at the top of the next page, where μ denotes the average electrical SNR of each OW link.

2) Independent and Not Identically Distributed OW Links: When the turbulence induced fading coefficients of the underlying optical links of the MIMO OW system are independent, but not identically distributed, the PDF of I_T can be approximated by

$$f_{I_T}(I) \approx \sum_{i=1}^{L}\sum_{j=1}^{m_i} W_L(i,j)\, f_\gamma\left(I; L, j, \frac{j\Omega_i}{m_i}\right) \qquad (33)$$

where $L = MN$ is the number of the underlying OW links, $m_l = a_{mn}$, $k_l = 1$ and $\Omega_l = \mathbf{E}[I_{mn}]$, when $m = 1, ...M$, $n = 1, ...N$ and $l = 1, ...MN$.

By substituting (33) to (29), the BER probability of the MIMO OW system is approximated by

$$P_e \approx \sum_{i=1}^{L}\sum_{j=1}^{m_i} W_L(i,j) \times \frac{1}{2}\int_0^\infty f_\gamma\left(I; L, j, \frac{j\Omega_i}{m_i}\right)\mathrm{erfc}\left(\frac{\eta}{2\sqrt{2}\mathrm{NM}\sigma_\upsilon}\mathrm{I}\right)\mathrm{dI} \qquad (34)$$

The integral in the above equation can be evaluated by expressing its integrands in terms of Meijer's G-function, as in the i.i.d. case. Hence, the probability of error is given by (35), at the top of the next page, where μ_i is the average electrical SNR of the ith OW link.

C. Outage Probability

The *outage probability* is defined as the probability that the SNR of the combined signal at the output of the receiver, h_T, falls below a specified threshold, h_{th}. It is considered as an important parameter for OW links to be operated as a part of a data network and is critical in the design of both transport and network layer. Hence, using (27), the resulting outage probability of the system is given by

$$P_{out} = \Pr(h_T \le h_{th}) = \Pr(I_T \le I_{th}) \qquad (36)$$

where $I_{th} = \frac{NM}{\eta}\sqrt{h_{th}N_o}$ is the normalized threshold.

1) Independent and Identically Distributed OW Links: When the underlying channels of the MIMO OW system are independent and identically distributed, the PDF of I_T can be approximated by the PDF of a single GG variate with parameters (k_T, m_T, Ω_T) as defined in Eq. (30). The

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$$P_e \approx \frac{2^{k_T+m_T-3}}{\sqrt{\pi^3}\Gamma\left(k_T\right)\Gamma\left(m_T\right)} G_{5,2}^{2,4}\left[\left(\frac{2}{k_T m_T}\right)^2 \mu \,\middle|\, \begin{matrix} \frac{1-k_T}{2}, \frac{2-k_T}{2}, \frac{1-m_T}{2}, \frac{2-m_T}{2}, 1 \\ 0, \frac{1}{2} \end{matrix}\right] \tag{32}$$

$$P_e \approx \sum_{i=1}^{L}\sum_{j=1}^{m_i} \frac{2^{L+j-3} W_L\left(i,j\right)}{\sqrt{\pi^3}\Gamma\left(L\right)\Gamma\left(j\right)} G_{5,2}^{2,4}\left[\left(\frac{2}{L^2 m_i}\right)^2 \mu_i \,\middle|\, \begin{matrix} \frac{1-L}{2}, \frac{2-L}{2}, \frac{1-j}{2}, \frac{2-j}{2}, 1 \\ 0, \frac{1}{2} \end{matrix}\right] \tag{35}$$

outage probability of the under consideration system can be approximated by

$$P_{out} \approx \int_0^{I_{th}} f_\gamma\left(I; k_T, m_T, \Omega_T\right) \mathrm{dI} \tag{37}$$

which is equivalent to the outage probability of a SISO system operating over the GG turbulence model with parameters (k_T, m_T, Ω_T). Hence, using (3), a closed-form expression for the outage probability yields as

$$P_{out} \approx \frac{1}{\Gamma\left(k_T\right)\Gamma\left(m_T\right)} G_{1,3}^{2,1}\left[\frac{k_T m_T}{\Omega_T} I_{th} \,\middle|\, \begin{matrix} 1 \\ k_T, m_T, 0 \end{matrix}\right] \tag{38}$$

2) Independent and Not Identically Distributed OW Links: When the underlying channels of the MIMO OW system are independent, but not identically distributed, the PDF of I_T can be approximated by a nested finite weighted sum of GG PDFs, according to Eq. (33). Hence, the outage probability of the under consideration system can be approximated by

$$P_{out} \approx \sum_{i=1}^{L}\sum_{j=1}^{m_i} W_L\left(i,j\right) \int_0^{I_{th}} f_\gamma\left(I; L, j, \frac{j\Omega_i}{m_i}\right) \mathrm{dI} \tag{39}$$

where $L = MN$, $m_l = a_{mn}$, $\Omega_l = \mathbf{E}\left[I_{mn}\right]$, $m = 1, ...M$, $n = 1, ...N$ and $l = 1, ...MN$. From the above equation, it is evident that the outage probability of the MIMO OW system can be approximated by a finite nested weighted sum of outage probabilities of SISO OW links operating over the GG turbulence model with parameters $\left(L, j, \frac{j\Omega_i}{m_i}\right)$. Using (3), an analytical expression is derived as

$$P_{out} \approx \sum_{i=1}^{L}\sum_{j=1}^{m_i} \frac{W_L\left(i,j\right)}{\Gamma\left(L\right)\Gamma\left(j\right)} G_{1,3}^{2,1}\left[\frac{L m_i}{\Omega_i} I_{th} \,\middle|\, \begin{matrix} 1 \\ L, j, 0 \end{matrix}\right] \tag{40}$$

D. Numerical Results

In Figs. 1 and 2, the BER and outage probability of MIMO OW systems operating over identically distributed strong turbulence channels with parameters $a = 4$ or $a = 10$ and $I_o = 1$, are depicted. Analytical results, using (32) and (38), are illustrated in comparison with Monte-Carlo simulations. It is observed that there is an excellent match between the approximation and the simulations in every SNR regime for both performance metrics. It is also clearly depicted that the derived approximative expressions are accurate for every MIMO deployment investigated, irrespective the number of transmit and/or receive apertures.

Figs. 3 and 4 depict the BER and outage probability of various MIMO deployments of transmit and receive apertures over

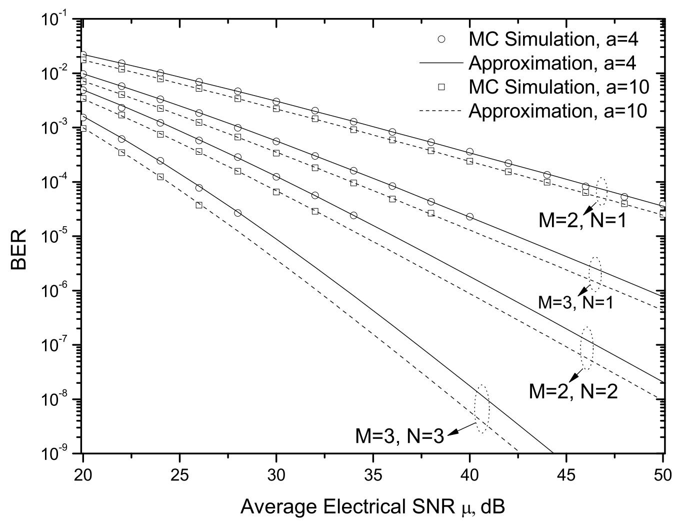

Fig. 1. Comparison of approximate average BER and MC simulation results for various MIMO OW systems over i.i.d. links

i.n.i.d. strong turbulence channels, i.e. the underlying OW links have different turbulence parameters and different average electrical SNRs. As it is clearly illustrated, the approximative analytical results for both performance metrics, derived from (35) and (40) respectively, are very close to the MC simulation results. Specifically, for the MIMO deployments investigated and for practical values of average BER and outage probability, the difference between analytical and simulation results is not greater than 2 dB. Moreover, it is observed that the proposed approximation acts as a lower bound, which becomes less accurate as the number of the underlying i.n.i.d. OW links increases. However, taking into consideration that the BER performance metric is difficult or even impossible to be evaluated with numerical techniques as the number of the OW links increases [17], the derived closed-form expressions can be considered as reliable alternatives to time consuming Monte Carlo simulations.

V. Conclusions

We examined the statistics of the sum of independent and not necessarily identical GG RVs. Novel closed-form expressions were derived that approximated its PDF either with the PDF of a single GG distribution, when all the variates of the sum were identically distributed, or with a finite weighted sum of PDFs of GG distributions, when the variates of the sum were non identically distributed. Based on the obtained

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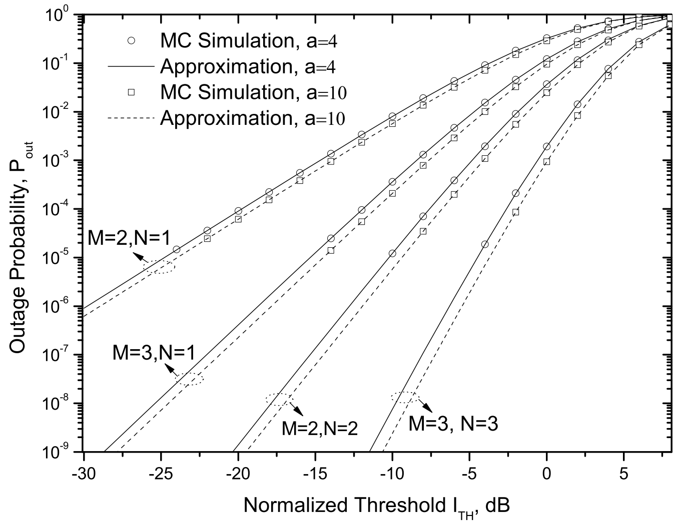

Fig. 2. Comparison of approximate outage probability and MC simulation results for various MIMO OW systems over i.i.d. links

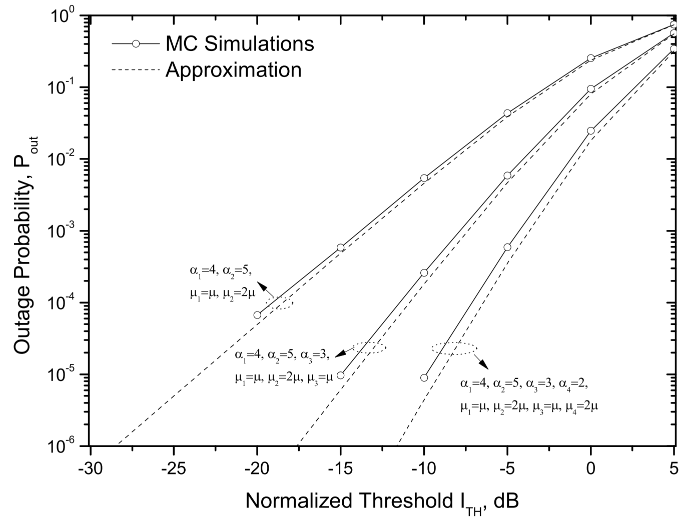

Fig. 4. Comparison of approximate outage probability and MC simulation results of MIMO systems with $M = 2$ and $N = 1$, $M = 3$ and $N = 1$, and $M = 2$ and $N = 2$ transmit and receive apertures.

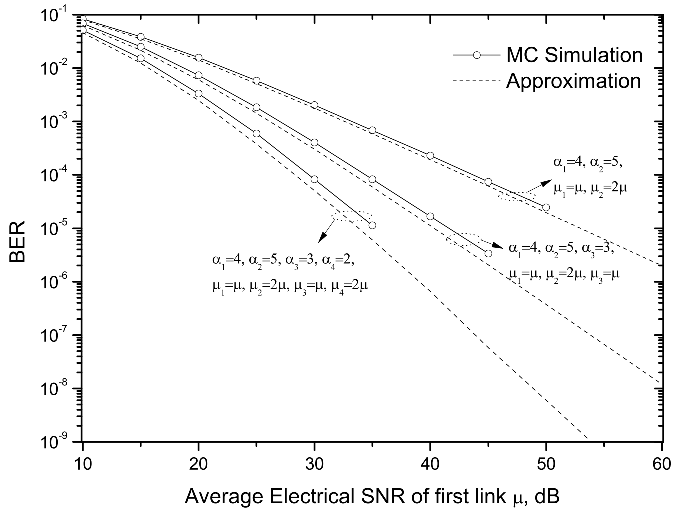

Fig. 3. Comparison of approximate BER performance and MC simulation results of MIMO systems with $M = 2$ and $N = 1$, $M = 3$ and $N = 1$, and $M = 2$ and $N = 2$ transmit and receive apertures.

statistical formulas, the performance of MIMO OW systems, operating over strong turbulence channels and employing EGC at the receiver was investigated and major performance metrics were analytically evaluated. The comparison between approximative analytical results and simulations demonstrated that the proposed approximation is accurate for identical underlying OW links, while it serves as a tight lower bound for non identical OW links.

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