ON THE SUM RATE OF ZF DETECTORS OVER CORRELATED K FADING MIMO CHANNELS

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ABSTRACT

This paper presents a detailed sum rate investigation of Zero-Forcing (ZF) detectors over composite multiple-input multiple-output (MIMO) channels. To this end, we consider the generic K distribution (Rayleigh/gamma distribution) to model the composite fading fluctuations and also assume the general case of semi-correlated small-scale fading. Novel exact analytical expressions are derived for the achievable sum rate followed by asymptotic expressions in the low Signal-to-Noise ratio (SNR) regime. In parallel, new, closed-form upper and lower bounds on the sum rate are derived that remain tight for all SNRs. The theoretical analysis is validated via a set of Monte-Carlo simulations.

1. INTRODUCTION

The capacity of point-to-point MIMO communication systems has been well investigated under Rayleigh fading conditions and assuming all different types of spatial correlation [1, 2]. However, little is still known for the capacity of distributed MIMO (D-MIMO) systems which exploit both spatial micro and macro-diversity [3–5]. The main feature of these systems is that multiple antennas at one end of the wireless channel are packed into multiple radio ports that are spatially separated. Hence, each link experiences different degree of path loss, as a result of the different access distances, along with different shadowing effects. Note that the latter manifestation is rather critical when assessing MIMO performance since it can significantly diminish the benefits of MIMO technology.

In this paper, we elaborate on the performance of ZF linear receivers over such composite D-MIMO channels. The fading fluctuations are modeled via the K-distribution [6, 7] which is a generic model that occurs when small-scale fading is modeled via the Rayleigh distribution and large-scale fading via the gamma distribution. This model has been demonstrated to effectively approximate most of the fading and shadowing effects occurring in wireless channels, and also to be analytically friendlier than the Rayleigh/lognormal model. Besides, this important implication was recently reported in [8], where the authors examined the achievable sum rate of ZF detectors over Rayleigh/lognormal MIMO channels. The final result, though, is given in integral form (see [8, Eq. (15)]) which was numerically approximated via Gauss-Hermite polynomials. This technique, however, is time-consuming (especially at low SNRs) and not amenable to further manipulations. In addition, the presented analysis was tied to co-located MIMO antenna arrays.

We hereafter pursue a statistical analysis of ZF detectors over composite K MIMO channels and also consider the more general case of correlated Rayleigh fading with correlation on the side with the minimum number of antennas. The exact achievable sum rate is analytically derived followed by a tractable affine expansion in the asymptotically low-SNR regime. In addition, we propose novel, closed-form upper and lower bounds on the sum rate. These expressions yield useful insights into the implications of the model parameters on the ZF detector's performance.

Notation: We use upper and lower case boldface to denote matrices and vectors, respectively. The $n \times n$ identity matrix reads as $\mathbf{I}_n$. The expectation is given by $\mathbb{E}[\cdot]$ while the symbols $(\cdot)^\dagger$ and $(\cdot)^H$ represent the pseudo-inverse and Hermitian transpose of a matrix, respectively.

2. MIMO SYSTEM MODEL

Let us consider a typical D-MIMO system equipped with N_r receive antennas and L radio ports each connected to N_t transmit antennas and also require that $LN_t \leq N_r$. In the following, we assume that the transmitter equally splits the available average power, P, amongst all data streams. Then, the input-output relationship reads as

$$\mathbf{y} = \sqrt{\frac{P}{LN_t}} \mathbf{H} \mathbf{\Xi}^{1/2} \mathbf{s} + \mathbf{n} \tag{1}$$

where $\mathbf{y} \in \mathbb{C}^{N_r \times 1}$ is the received signal vector, $\mathbf{s} \in \mathbb{C}^{LN_t \times 1}$ is the vector containing the transmitted symbols which are drawn from a unit-power constellation while the complex AWGN term is zero-mean with covariance $\mathbb{E}\left[\mathbf{n}\mathbf{n}^H\right] = N_0 \mathbf{I}_{N_r}$. The diagonal matrix $\mathbf{\Xi} \in \mathbb{R}^{LN_t \times LN_t}$ represents the large-scale fading, and hence $\boldsymbol{\Xi} = \operatorname{diag}\{\mathbf{I}_{N_t}\xi_m/D_m^{\upsilon}\}_{m=1}^{L}$ where D_m denotes the distance between the receiver and the m-th radio port while υ is the path-loss exponent. The large-scale fading coefficients $\xi_m, m = 1, \ldots, L$, are modeled as gamma random variables (RVs), or

$$f_{\xi_m}(\xi_m) = \frac{\xi_m^{k_m-1}}{\Gamma(k_m)\Omega_m^{k_m}} \exp\left(-\frac{\xi_m}{\Omega_m}\right), \quad \xi_m, \Omega_m, k_m \geq 0 \tag{2}$$

where k_i, $\Omega_i = \mathbb{E}[\xi_i]/k_i$, are the so-called shape and scale parameters of the gamma distribution respectively while $\Gamma(\cdot)$ is the well-known gamma function. As was previously mentioned, the small-scale fading is assumed to follow a semi-correlated Rayleigh distribution and as such, we can express $\mathbf{H} \in \mathbb{C}^{N_r \times LN_t}$ according to $\mathbf{H} = \mathbf{H}_w \mathbf{R}_t^{1/2}$ where the entries of $\mathbf{H}_w$ are modeled as i.i.d. $\mathcal{CN}(0,1)$ RVs while $\mathbf{R}_t$ is the positive definite covariance matrix of every row of $\mathbf{H}$.

After defining $\mathbf{Z} \triangleq \mathbf{H}\boldsymbol{\Xi}^{1/2}$, the ZF filter matrix becomes $\mathbf{G} = (P/LN_t)^{-1/2}\mathbf{Z}^{\dagger}$ [9]. Assuming independent decoding, the received signal is decomposed into LN_t parallel streams with the instantaneous received SNR at the m-th ZF output ($1 \leq m \leq LN_t$) being

$$\gamma_m \triangleq \frac{\rho}{LN_t\left[\left(\mathbf{Z}^H\mathbf{Z}\right)^{-1}\right]_{mm}} = \frac{\rho\left[\boldsymbol{\Xi}\right]_{mm}}{LN_t\left[\left(\mathbf{H}^H\mathbf{H}\right)^{-1}\right]_{mm}} \tag{3}$$

where $\rho = P/N_0$ is the average SNR and $[\cdot]_{mm}$ returns the m-th diagonal element of a matrix. We can also define the random small-scale counterpart as follows

$$x_m = \frac{1}{\left[\left(\mathbf{H}^H\mathbf{H}\right)^{-1}\right]_{mm}} \tag{4}$$

which follows a complex semi-correlated central Wishart distribution $x_m \sim \mathcal{CW}(N_r - LN_t + 1, 1/[\mathbf{R}_t^{-1}]_{mm})$ and its PDF was explicitly given in [10, Theorem 1]

$$f_{x_m}(x_m) = \frac{\sigma_m e^{-x_m\sigma_m}}{(N_r - LN_t)!}\left(x_m\sigma_m\right)^{N_r - LN_t} \tag{5}$$

where σ_m is the m-th diagonal entry of $\mathbf{R}_t^{-1}$.

3. ACHIEVABLE SUM RATE OF ZF DETECTORS

3.1. Exact analysis

The maximum achievable sum rate, assuming independent decoding at the receiver, is essentially the summation of the throughputs offered by each subchannel, or

$$R \triangleq \sum_{m=1}^{LN_t} \mathbb{E}\left[\log_2(1+\gamma_m)\right] = \sum_{m=1}^{LN_t} \mathbb{E}\left[\log_2\left(1 + \frac{\rho\xi_m x_m}{LN_t D_m^{\upsilon}}\right)\right] \tag{6}$$

where the expectation is taken over all realizations of $\mathbf{H}$ and $\boldsymbol{\Xi}$. The following theorem returns an analytical expression for the exact sum rate of ZF detectors:

Theorem 1 *The achievable sum rate of ZF detectors over correlated K MIMO channels is given by*

$$R = \frac{1}{\ln 2 (N_r - LN_t)!} \sum_{m=1}^{LN_t} \frac{1}{\Gamma(k_m)} \times G_{4,2}^{1,4}\left[\frac{\rho\Omega_m}{LN_t\sigma_m D_m^{\upsilon}} \,\middle|\, \begin{matrix} 1-k_m, LN_t - N_r, 1, 1 \\ 1, 0 \end{matrix}\right] \tag{7}$$

where $G_{p,q}^{m,n}\left[x, \middle|\begin{smallmatrix}\alpha_1,\ldots,\alpha_p \\ \beta_1,\ldots,\beta_q\end{smallmatrix}\right]$ is the Meijer's G-function [11, Eq. (9.301)].

Proof: The proof starts by expressing the expectation in (6) in integral form as

$$R = \frac{1}{\ln 2}\sum_{m=1}^{LN_t}\int_0^{\infty}\int_0^{\infty}\ln\left(1 + \frac{\rho\xi_m x_m}{LN_t D_m^{\upsilon}}\right) \times f_{\xi_m}(\xi_m) f_{x_m}(x_m) d\xi_m dx_m. \tag{8}$$

Substituting (2) and (5) into (8) and successively applying [11, Eq. (7.813.1)], we can obtain (7) after some basic algebraic manipulations.

Note that at high-SNRs, the sum rate converges to

$$R^{\infty} = LN_t \log_2\left(\frac{\rho}{LN_t}\right) + \frac{LN_t}{\ln 2}\psi(N_r - LN_t + 1) - \sum_{m=1}^{LN_t}\log_2\left(\sigma_m\right) + N_t\sum_{m=1}^{L}\left(\frac{\psi(k_m)}{\ln 2} + \log_2\left(\frac{\Omega_m}{D_m^{\upsilon}}\right)\right).$$

where $\psi(x)$ is Euler's digamma function [11, Eq. (8.360.1)]. The above expression is quite intuitive since it indicates that at high SNRs the effects of small and large-scale fading are decoupled. We also validate the diminishing effects of spatial correlation on the sum rate since, due to Hadamard's inequality, $1 \leq \det\left(\mathbf{R}_t^{-1}\right) \leq \prod_{m=1}^{LN_t}\sigma_m$, with equality for $\mathbf{R}_t = \mathbf{I}_{LN_t}$. Clearly, larger values of k_m, Ω_m diminish the effects of shadowing thereby delivering higher sum rate while longer terminal distances, D_m, lead to stronger path-loss attenuation.

3.2. Low-SNR analysis

It has been theoretically shown that the low-SNR sum rate can be more effectively explored in terms of the normalized transmit energy per information bit E_b/N_0 rather than per-symbol SNR, or [12, 13]

$$R\left(\frac{E_b}{N_0}\right) \approx \mathcal{S}_0 \log_2\left(\frac{\frac{E_b}{N_0}}{\frac{E_b}{N_0}_{\min}}\right) \tag{9}$$

where $\frac{E_b}{N_0}_{\min}$ and $\mathcal{S}_0$ correspond to the "minimum normalized energy per information bit required to convey any positive rate reliably" and the wideband slope, respectively.

Corollary 1 *The minimum energy per information bit and the wideband slope of ZF detectors over correlated K MIMO channels are respectively*

$$\frac{E_b}{N_{0\,min}} = \frac{LN_t \ln 2}{(N_r - LN_t + 1)} \left(\sum_{m=1}^{LN_t} \frac{k_m \Omega_m}{\sigma_m D_m^{\upsilon}} \right)^{-1}$$

$$\mathcal{S}_0 = \frac{2(N_r - LN_t + 1)}{(N_r - LN_t + 2)} \frac{\left(\sum_{m=1}^{LN_t} \frac{k_m \Omega_m}{\sigma_m D_m^{\upsilon}} \right)^2}{\sum_{m=1}^{LN_t} k_m (k_m + 1) \left(\frac{\Omega_m}{\sigma_m D_m^{\upsilon}} \right)^2}.$$

Proof: From [13], we have that $\frac{E_b}{N_{0\,\min}}$ and $\mathcal{S}_0$ are respectively

$$\frac{E_b}{N_{0\,\min}} \triangleq \lim_{\rho \to 0} \frac{\rho}{R(\rho)} = \frac{1}{\dot{R}(0)} \quad \text{and} \quad \mathcal{S}_0 \triangleq -\frac{2 \ln 2 \left[\dot{R}(0) \right]^2}{\ddot{R}(0)}$$

where $\dot{R}(\cdot)$ and $\ddot{R}(\cdot)$ denote the first and second-order derivatives of the sum rate (6) over the SNR ρ. The desired results follow easily after appropriate simplifications.

For i.i.d. Rayleigh fading, it can be shown that $\frac{E_b}{N_{0\,\min}} = \ln 2/(N_r - N_t + 1)$ and $\mathcal{S}_0 = 2N_t(N_r - N_t + 1)/(N_r - N_t + 2)$, which are respectively identical with [15, Eq. (25)] and [15, Eq. (26)]. In the specific case of $N_t = 1$, the slope particularizes to $\mathcal{S}_0 = 2N_r/(N_r + 1)$ which coincides with the associated results for both MMSE [14, Eq. (52)], [15, Eq. (24)] and optimal receivers [12, Eq. (16)], thereby validating that at low SNRs, ZF detection becomes optimal for a single transmit antenna since all N_r degrees of freedom are devoted to the recovery of the corresponding multiplexed stream. All these results further demonstrate that for a fixed N_r, increasing N_t may improve the wideband slope, but also increases the minimum energy per bit due to the additional power required to cancel out the extra interferers.

3.3. Tight bounds on the sum rate

Since the expression in Theorem 1 is given via a Meijer-G function, which is a general function with a hard practical interpretation, it is important to obtain tight upper and lower closed-form bounds on the achievable sum rate.

Theorem 2 *The achievable sum rate of ZF detectors over correlated K MIMO channels in (6) is upper bounded by*

$$R_U = \sum_{m=1}^{LN_t} \log_2 \left(1 + \frac{\rho(N_r - LN_t + 1)}{LN_t} \frac{k_m \Omega_m}{\sigma_m D_m^{\upsilon}} \right). \quad (10)$$

Proof: The proof follows by applying Jensen's inequality on (6) along with the methodology of [16, Theorem 4].

Corollary 2 *In the high-SNR regime, the upper bound in (10) becomes*

$$R_U^{\infty} = LN_t \log_2 \left(\frac{\rho}{LN_t} \right) + LN_t \log_2 (N_r - LN_t + 1) - \sum_{m=1}^{LN_t} \log_2 (\sigma_m) + N_t \sum_{m=1}^{L} \log_2 \left(\frac{k_m \Omega_m}{D_m^{\upsilon}} \right). \quad (11)$$

Interestingly, the upper bound becomes tighter for higher N_r, k_m. In the limit, $N_r, k_m \to \infty$, the bound becomes exact since $\psi(x) \approx \ln(x)$ if $x \to \infty$. On a similar basis, we can now propose the following new lower bound on the sum rate, which becomes exact at high-SNRs.

Theorem 3 *The achievable sum rate of ZF detectors over correlated K MIMO channels in (6) is lower bounded by*

$$R_L = \sum_{m=1}^{LN_t} \log_2 \left(1 + \frac{\rho}{LN_t} \exp \left(\psi (N_r - LN_t + 1) + \psi (k_m) + \ln \left(\frac{\Omega_m}{\sigma_m D_m^{\upsilon}} \right) \right) \right). \quad (12)$$

Proof: The proof relies on the generic bounding technique of [2, Theorem 1] which is omitted for the sake of brevity. A detailed proof is given in a journal version of this paper [16].

4. NUMERICAL RESULTS

We first generate 20,000 random realizations of the small and large-scale fading matrices $\mathbf{H}$ and $\mathbf{\Xi}$ according to (2), and thereafter obtain the simulated sum rate via (6). The transmit correlation matrix is constructed as $\mathbf{R}_t = \text{diag}\{\mathbf{R}_{t,m}\}_{m=1}^{L}$ where $\mathbf{R}_{t,m}$ is the correlation matrix between the antennas of the m-th terminal. The entries of the latter are modeled via the common exponential correlation model $\{\mathbf{R}_{t,m}\}_{i,j} = \rho_{t,m}^{|i-j|}$ with $\rho_{t,m} \in [0, 1)$ being the transmit correlation coefficient.

In Fig. 1, the achievable sum rate is plotted vs the SNR, ρ. For the sake of simplicity, we have set $k_m = 1, \Omega_m = 2, \rho_{t,m} = 0.3$, $\forall m = 1, \ldots, LN_t$ while we consider different MIMO configurations by keeping $N_t = 2$ and increasing only N_r. The outputs of a Monte-Carlo simulator are compared with the exact analytical expression of Theorem 1 and the upper/lower bounds of (10) and (12), respectively. It is easily observed that the match between theory and simulation is excellent in all cases under consideration, thereby validating the correctness of the proposed analytical expressions. The graph indicates that adding more receive antennas significantly stabilizes the MIMO link by improving the receive diversity and reducing the noise enhancement effect. This is consistent with the results in [9]. Further, a higher N_r makes both bounds tighter although the upper bound is in general less tight due to the loose nature of Jensen's inequality. As previously mentioned, at high SNRs the lower bound becomes exact.

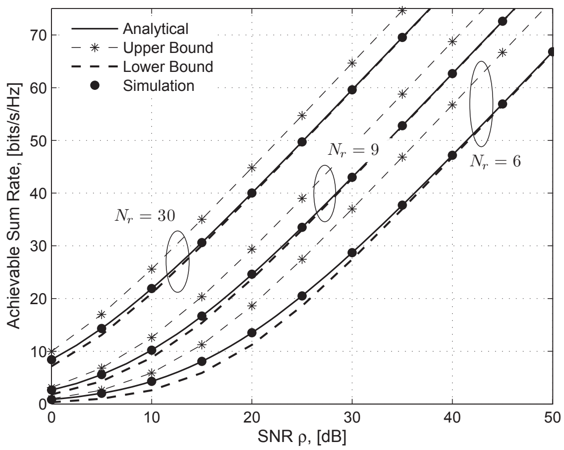

Fig. 1. Simulated sum rate, analytical expression and upper/lower bounds against the SNR ($L = 3$, $\upsilon = 4$, $D_1 = 1000$ m, $D_2 = 1500$ m, $D_3 = 2000$ m).

In Fig. 2, the analytical and simulated low-SNR sum rate are depicted against the transmit energy per bit E_b/N_0 (Corollary 1). For illustration purposes, we consider only small-scale fading (both correlated and i.i.d. Rayleigh fading). It can be observed that increasing N_t increases the minimum energy per bit and so does the correlation coefficient ρ_t. The wideband slope is enhanced with a higher N_t and seems to remain relatively unaffected by spatial correlation. For all scenarios, it turns out that the linear approximations are quite accurate over a moderate range of E_b/N_0 values.

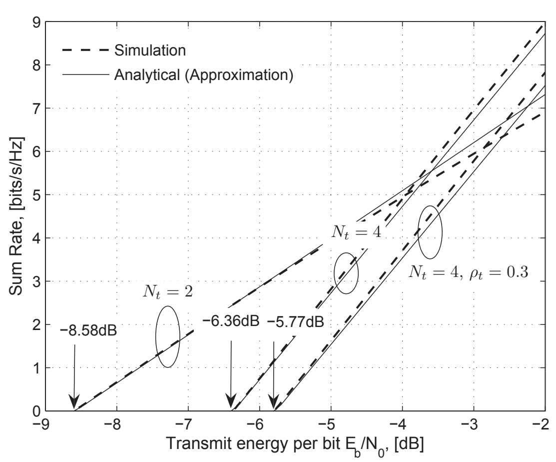

Fig. 2. Low-SNR simulated sum and analytical linear approximation against the transmit energy per bit ($N_r = 6$).

5. CONCLUSION

The main focus of this paper was on the sum rate characterization of ZF detectors over correlated K MIMO channels. To this end, a novel analytical formula for the exact maximum achievable sum rate was provided along with a tractable affine expansion in the asymptotically low-SNR regime. Simple closed-form upper and lower bounds were also provided, which were demonstrated to remain tight across the entire SNR range. We finally explored via numerical simulations, the implications of the model parameters on the performance of ZF detectors over composite K fading MIMO channels.

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