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On the Inverse-Gaussian Shadowing

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***Abstract*—The Inverse Gaussian (IG) distribution has been recently proposed as a less complex alternative to the classical log-normal distribution to describe shadowing phenomena. This paper investigates the performance of digital communications systems over IG fading channels. Closed-form expressions for the outage probability and the average bit-error rate for various modulation schemes are derived. Moreover the performance of maximal-ratio and selection combining receivers is analyzed. The proposed analysis is accompanied with various numerical results that validate the theoretical analysis and provide useful insights into the implications of the model parameters on the overall system performance.**

I. Introduction

A lot of experimental measurements have shown that the log-normal (LN) distribution is the sovereign model to describe the shadowing phenomenon due to large obstructions on outdoor, in-door and satellite channels [1]. Particularly, in slowly varying channels, the small- and large-scale effects get mixed. and the log-normal statistics tend to accurately describe the distribution of the channel path gain [2, ch. 9.13]. In such a model, the probability density function (pdf) of the signal-to-noise (SNR) ratio per symbol, γ, is given by [2, eq. (2.53)]

$$f_\gamma(\gamma) = \frac{\xi}{\sqrt{2\pi}\sigma\gamma} \exp\left[-\frac{(10\log_{10}\gamma - \mu)^2}{2\sigma^2}\right], \quad \gamma > 0 \quad (1)$$

where $\xi = \frac{10}{\ln 10}$, $\mu\,(dB)$ is the mean and $\sigma\,(dB)$ the standard deviation of $10\log_{10}\gamma$, respectively.

Despite its extensive use, the LN model is analytically intractable when it is used for the performance evaluation of digital communications systems over LN fading channels. This stems from the fact that the expression for the moment-generating function (MGF) of the LN distribution cannot be derived in closed-form. Hence, no closed-form expressions are available in the open technical literature for the error performance, while the ergodic capacity of LN channels is also provided using approximations [3]. Furthermore, the distribution of a sum of LN random variables (RVs)-an old problem- is not known in closed form and is difficult to compute it numerically. Therefore, approximations are also used in order to study the performance of diversity receivers [2].

Things are getting more complicated when composite multipath/shadowed models are taken into account. For example, the well known Rayleigh-LN (RLD) model does not have a closed-form expression. In a way to find simpler composite models, the gamma distribution was proposed as a substitute to the LN and the Rayleigh-Gamma, known also as K distribution appeared in the literature [4]. Recently, the Inverse Gaussian (IG) distribution (also known as Wald distribution) has been proposed as an alternative substitute for the LN one. Karmeshu and Agrawal in [5] used the Kullback-Leibler found that the IG approximates better the LN distribution than the Gamma does. In this work, the authors introduced the Rayleigh-IG as a new effective composite model.

Motivated by the problems raised in communications theory, where the LN distribution is involved, we introduce the Inverse Gaussian (IG) distribution as a statistical model, which can be efficiently used to describe shadowing in wireless communications, providing simplicity as well. IG distribution was proposed by M. C. K. Tweedie in 1945 during a statistical investigation of data relating to electrophoretic measurements. He observed that encountered distribution possessed a remarkable property; namely, its cumulant generating function turned out to be the inverse of that of the Gaussian distribution. Note, that Tweedie also noted this type of relationship between the binomial and the negative binomial, and between the Poisson and the exponential distribution. The physicists and probabilists call this the first passage time distribution of Brownian motion with drift [6]. Later, this statistical model was introduced in optical systems, to model the statistical behavior of avalanche photo diodes receivers [7].

In this paper, we study the performance of wireless communications systems operating over IG fading channels. More specifically, we present closed-form expressions for the outage probability and the average symbol error rate of single channel reception for a variety of modulation formats. Furthermore, capitalizing on these results, novel analytical expressions for the performance of Maximal Ratio Combining (MRC) and Selection Combining (SC) receivers are derived. To the best of the authors' knowledge, such an analysis is novel. Various selected numerical results and comparisons with the LN-based model are also provided.

The remainder of the paper is organized as follows: In Section II, we present the IG distribution and introduce its basic statistical parameters. In Section III, we present closed-form expressions for the outage probability and the average BER of a point-to-point communication link, which operates in IG shadowing. In Section IV the performance of MRC and SC diversity receivers are studied when IG fading is assumed. In the same section, we compare the distribution of the sum of IG RVs with the corresponding one of the LN model. Finally, concluding remarks are given in Section V.

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II. Statistical properties of the IG Distribution

In the IG model, the instantaneous SNR has the following pdf [8, eq. (27.1.1)]

$$f_\gamma(\gamma) = \sqrt{\frac{\lambda}{2\pi\gamma^3}} \exp\left(-\frac{\lambda(\gamma-\theta)^2}{2\vartheta^2\gamma}\right), \quad \gamma > 0 \quad (2)$$

where $\theta > 0$ is the parameter related to the mean of the fluctuations and $\lambda > 0$ is the scale parameter of the distribution. This is also denoted as $\gamma \sim IG(\theta, \lambda)$.

The n-th moment of the output SNR is found to be given by [8, eq. (27.2)]

$$E[\gamma^n] = \theta^n \sum_{k=0}^{n-1} \frac{(n-1+k)!}{k!(n-1-k)!}\left(2\frac{\lambda}{\theta}\right)^{-k}, \quad (3)$$

where $E[\cdot]$ denotes expectation. Therefore, the average SNR is obtained as

$$\overline{\gamma} = E[\gamma] = \theta. \quad (4)$$

The parameters of the IG distribution, λ and θ, can be related to the LN distribution ones, μ and σ, by matching the first and second moments of the two distributions, respectively, leading to

$$\begin{cases} \lambda = \frac{\exp\left(\frac{\mu}{\xi}\right)}{2\sinh\left(\frac{\sigma^2}{2\xi^2}\right)} \\ \theta = \exp\left(\frac{\mu}{\xi} + \frac{\sigma^2}{2\xi^2}\right) \end{cases} \quad (5)$$

where $\xi = \frac{10}{\ln 10}$.

Moreover, using (3), we can determine the amount of fading (AF) which is a unified measure of the severity of fading, as

$$AF = \frac{E[\gamma^2]}{E[\gamma]^2} - 1 = \frac{\theta}{\lambda}, \quad (6)$$

which, after some algebra, yields to

$$AF = \exp\left(\frac{\sigma^2}{\xi^2}\right) - 1, \quad (7)$$

and equals to the corresponding AF for LN model [2, Eq. (2.56)].

III. Performance of Single Channel Receivers

A. Outage Probability

Outage probability is an important performance metric of wireless communications systems. It is defined as the probability that the instantaneous SNR falls below a specified threshold, γ_{th}, and represents a protection value of the SNR above which the quality of the channel is satisfactory. It is defined as

$$P_{out} \triangleq \Pr(\gamma \le \gamma_{th}) = F_\gamma(\gamma_{th}), \quad (8)$$

where $F_\gamma(\cdot)$ is the cumulative distribution function (cdf) of the instantaneous SNR.

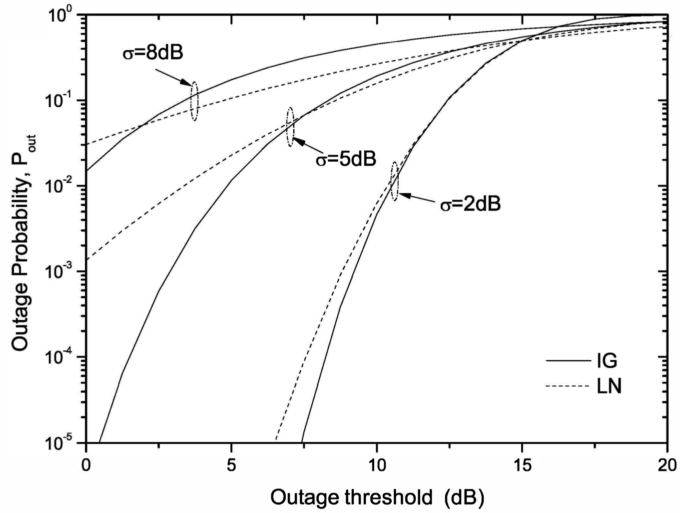

Fig. 1. Outage probability vs outage threshold for various standard deviations and $\mu = 15dB$.

The cdf for the IG model can be expressed in terms of the Gaussian Q-function, using the expression provided in [8, eq. (27.1.2)], as

$$F_\gamma(\gamma) = Q\left(\sqrt{\frac{\lambda}{\gamma}}\left(1-\frac{\gamma}{\theta}\right)\right) + e^{\frac{2\lambda}{\theta}} Q\left(\sqrt{\frac{\lambda}{\gamma}}\left(\frac{\gamma}{\theta}+1\right)\right), \quad (9)$$

where $Q(\cdot)$ is the Gaussian Q-function, defined as $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{t^2}{2}}dt$ and related to the complementary error function erfc$(\cdot)$ by $Q(x) = \frac{1}{2}\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$.

Assuming LN model, the cdf of eq. (1), is given by [2, eq. (2.56)]

$$F_\gamma(\gamma) = Q\left(\frac{\mu - 10\log_{10}(\gamma)}{\sigma}\right). \quad (10)$$

Fig. 1 gives a better understanding as to how shadowing affects the outage probability. We consider a typical value of mean of $\mu = 15dB$ and we give plots assuming IG and LN models for different shadowing levels (different standard deviations). It is clearly observed that the two cdfs for a low standard deviation value approximate each other; however, the accuracy of the approximation looses as the shadowing severity increases.

B. Average Symbol Error Rate

The average symbole error rate (ASER) for various modulation schemes can be evaluated using the MGF-based approach [2]. As an example, for non-coherent detection of binary orthogonal frequency-shift-keying (FSK) and for differentially coherent detection of binary phase-shift-keying (DPSK) the average BER is given by

$$P_b(e) = A_1 M(-a_1), \quad (11)$$

where $M(\cdot)$ is the MGF and A_1 and α_1 are constants depending on the modulation scheme. The MGF of the instantaneous

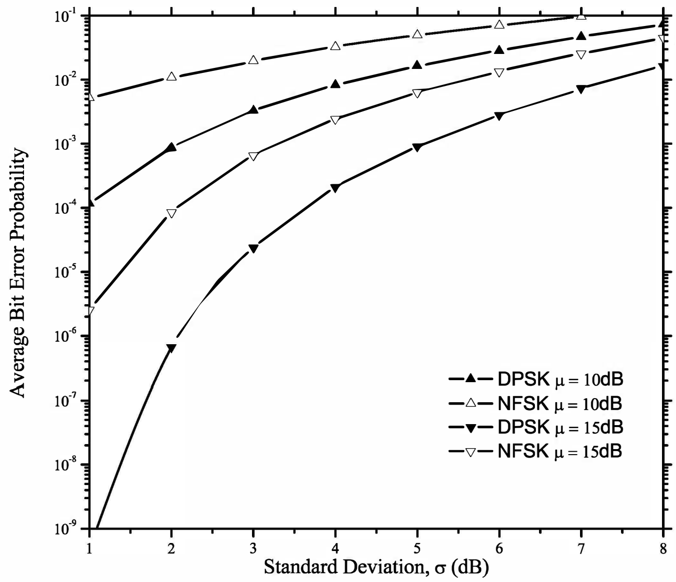

Fig. 2. Average Bit Error Probability vs standard deviation for DPSK and NFSK modulations.

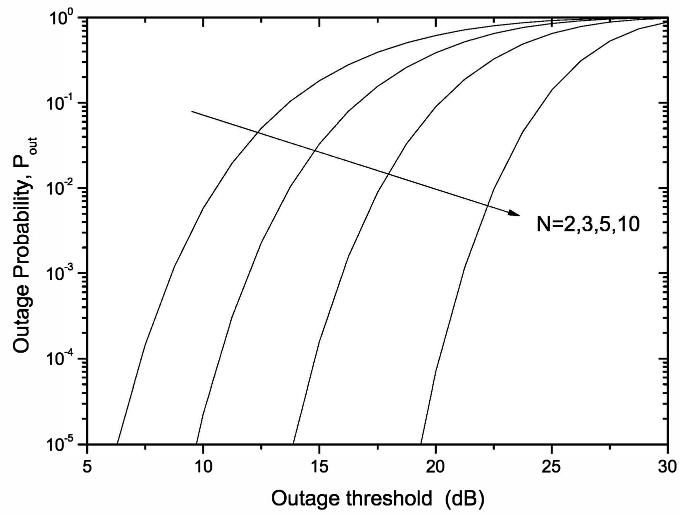

Fig. 3. Outage probability of MRC receivers vs outage threshold for $\mu = 15dB$ and $\sigma = 5dB$.

SNR for the IG distribution is given in closed form as [8, eq. (27.2)]

$$M_\gamma(s) = e^{\frac{\lambda}{\theta}\left[1-\sqrt{1-\frac{2\theta^2 s}{\lambda}}\right]}. \tag{12}$$

The average BER for DPSK and NFSK modulations for various shadowing levels is illustrated in Fig. 2 in terms of some typical standard deviation values. These values are met in practical wireless systems [1]. For DPSK $A_1 = \frac{1}{2}$, $\alpha_1 = 1$, and for NFSK $A_1 = \frac{1}{2}$, $\alpha_1 = \frac{1}{2}$. It is clearly shown that the shadowing severity degrades the BER.

IV. Performance Of Diversity Receivers

A. Maximal ratio Combining (MRC)

For equally likely transmitted symbols, the total SNR per symbol γ_{MRC} at the output of a N-branch MRC receiver equals to the sum of the individual diversity branches SNRs, i.e. [2, eq. (2.53)]

$$\gamma_{MRC} = \sum_{i=1}^{N} \gamma_i. \tag{13}$$

Assuming $\gamma_i \sim IG(\theta,\ \lambda)$, it can be easily found according to [9, ch. 2.4.2] that

$$\gamma_{MRC} \sim IG\left(\sum \theta_i,\ \xi\left(\sum \theta_i\right)^2\right) \tag{14}$$

if and only if $\xi = \lambda_i/\theta_i^2$ for all i. In the case of independent and identically distributed (i.i.d.) individual SNRs ($\gamma = \gamma_i$ for all i), the above restriction is satisfied and the total SNR is IG distributed as

$$\gamma_{MRC} \sim IG\left(N\theta,\ N^2\lambda\right), \tag{15}$$

where ($\theta = \theta_i$ and $\lambda = \lambda_i$ for all i).

By having the total SNR in closed form, it is easy to derive analytically the outage probability of an MRC receiver. As an example, Fig. 3 shows the outage probability of $N = 2, 3, 5, 10$ received antennae versus the outage threshold, where we observe an improved performance as the number of branches increases.

Meanwhile, the MGF of the γ_{MRC} can easily be obtained as

$$M_{\gamma_{MRC}}(s) = \prod_{i=1}^{N} M_{\gamma_l}(s), \tag{16}$$

where $M_{\gamma_l}(\cdot)$ is given in (12). For i.i.d SNRs the above equation becomes

$$M_{\gamma_{MRC}}(s) = \left[M_\gamma(s)\right]^N. \tag{17}$$

In Fig. 4, the average BER performance of DPSK and NFSK for $\mu = 10dB$ with $N = 2, 3$ and 5 antennae is depicted. We notice that a target BER $= 10^{-8}$ is possible to be achieved for $\sigma = 5dB$ by assuming DPSK modulations and 5 received antennae. Of course, a more increased BER performance is expected for a greater number of diversity branches.

B. Comparison with the LN Model

In the following, using statistical tools we demonstrate that the distribution of the sum of i.i.d IG RVs, given by 15, can serve as a more tractable alternative to the LN distribution, where a corresponding closed-form is unknown. Specifically, we employ Kolmogorov-Smirnov (KS) goodness of fit statistical tests in order to measure the difference between the two distributions [10, pp. 272-273]. Hence, for the case under consideration, the KS test statistic is defined as

$$T \triangleq \max\left|F_{\gamma_{MRC}}(x) - F_y(x)\right|, \tag{18}$$

where $F_{\gamma_{MRC}}(\cdot)$ is the CDF of the sum of L i.i.d. IG RVs analytically evaluated using (9), (15) and $F_y(\cdot)$ is the cdf of

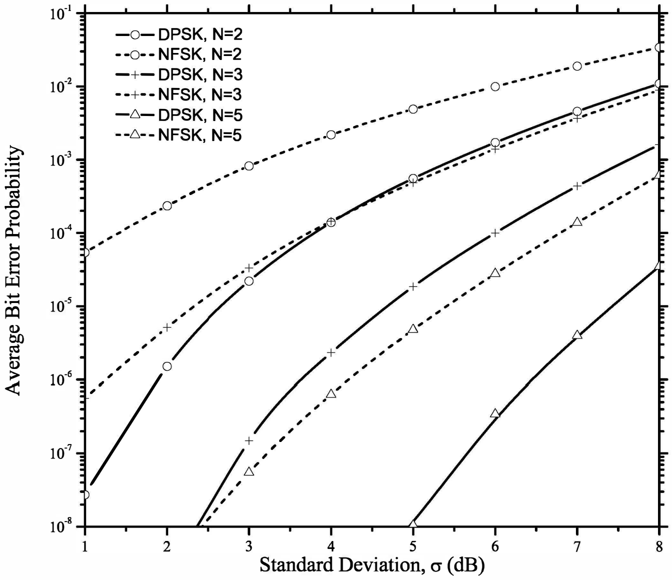

Fig. 4. Average Bit Error Probability of MRC receivers vs standard deviation for DPSK and NFSK modulations ($\mu = 10dB$).

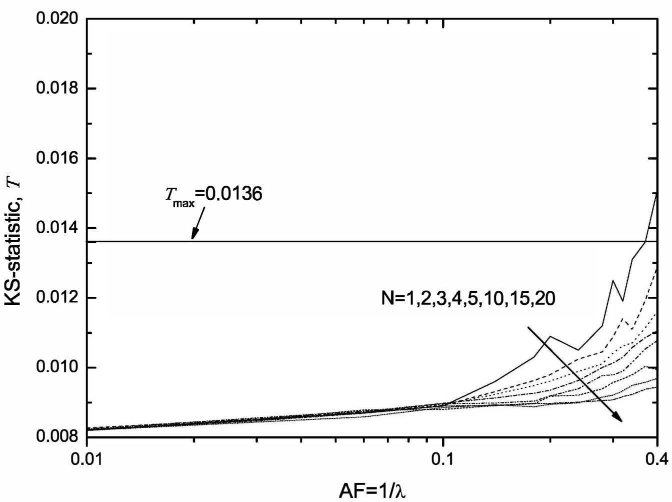

Fig. 5. Hypothesis testing distribution using the KS goodness-of-fit test for the IG to approximate the LN distribution with a 5% significance level.

the sum of N i.i.d. LN RVs, which is produced via Monte Carlo simulations, since there is no closed form expression when $N > 1$.

Definition 1: We define $\boldsymbol{H}_0$ as the null hypothesis under which the IG distributed data belongs to the cdf of the LN distribution $F_y(\cdot)$.

To test $\boldsymbol{H}_0$, the KS goodness-of-fit test compares the test statistic T to a critical level $T_{\max}$ for a given significance level α. Any hypothesis for which $T > T_{\max}$, is rejected with significance $1-\alpha$, while any hypothesis for which $T < T_{\max}$ is accepted with the same level of significance.

Fig. 5 depicts the KS test statistic for different values of L and $AF = \frac{1}{\lambda}$, where it is assumed that $\theta = 1$. The presented test results have been obtained by averaging the results of 60 simulation runs, each for at least 10^4 samples of IG distributed

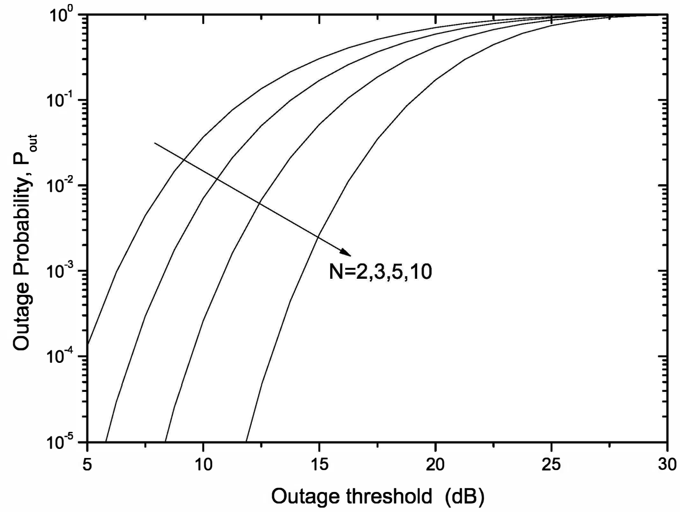

Fig. 6. Outage probability of SC receivers vs outage threshold for $\mu = 15dB$ and $\sigma = 5dB$.

data. The critical value $T_{\max}$, used for comparison reasons, is equal to $T_{\max} = 0.0136$ when a significance level of $\alpha = 5\%$ is considered [10, Eq. (9-73)]. It is clearly illustrated that for $(L = 1)$, the hypothesis $\boldsymbol{H}_0$ is accepted with 95% significance until $\frac{1}{\lambda}$ reaches the value of 0.35. Furthermore, when increasing N, it is observed that the KS statistic is reduced, indicating that the accuracy of the IG model is improved.

C. Selection Combining (SC)

It is well known that the cdf of the output SNR, γ_{SC}, when N branches SC is performed at the receiver is given by,

$$F_{SC}(x) = [F_\gamma(x)]^N \tag{19}$$

assuming i.i.d IG fading channel. Using (9) and the binomial theorem [11, Eq. (1.111)], (19) is rewritten as

$$\begin{aligned} F_{SC}(x) &= \sum_{k=0}^{N} \frac{N! \exp\left(\frac{2\lambda(N-k)}{\theta}\right)}{k!\,(N-k)!} Q^k\left(\sqrt{\frac{\lambda}{\gamma}}\left(1-\frac{\gamma}{\theta}\right)\right) \\ &\quad \times Q^{N-k}\left(\sqrt{\frac{\lambda}{\gamma}}\left(\frac{\gamma}{\theta}+1\right)\right). \end{aligned} \tag{20}$$

Note, that the powers of the $Q-$function can be efficiently evaluated using the approach proposed in [12].

Fig. 6 shows the outage probability of SC scheme, when $N = 2, 3, 5, 10$ received antennae is assumed, versus the outage threshold. Although less optimum than MRC, we observe an improved performance as the number of branches increases.

V. Conclusions

We have studied the performance of wireless communications systems operating over IG fading channels. After deriving closed-form expressions for the outage probability and the average symbol error rate of single channel reception for a variety of modulation formats, we investigated the performance of Maximal Ratio Combining (MRC) and Selection Combining (SC) receivers are derived. Numerical results and comparisons with the LN-based model have been also presented which validated the accuracy of the theoretical analysis.

REFERENCES

[1] G. L. Stuber, *Principles of Mobile Communication*, 2nd ed. New York: Kluwer, 2001.

[2] M. K. Simon and M.-S. Alouini, *Digital Communication over Fading Channels*, 2nd ed. New York: Wiley, 2005.

[3] F. Héliot, X. Chu, R. Hoshyar, and R. Tafazolli, "A tight closed-form approximation of the Log-Normal fading channel capacity," *IEEE Trans. Wireless Commun.*, vol. 8, no. 6, pp. 2842–2847, Jun. 2009.

[4] A. Abdi and M. Kaveh, "K distribution: An appropriate substitute for Rayleigh-Lognormal distribution in fading-shadowing wireless channels," *Electron. Lett*, vol. 34, no. 9, pp. 851–852, Apr. 1998.

[5] Karmeshu and R. Agrawal, "On efficacy of Rayleigh-Inverse Gaussian distribution over K-distribution for wireless fading channels," *Wireless Commun. Mobile Comput.*, vol. 7, no. 1, pp. 1–7, Jan. 2007.

[6] B. Seshardi, *The Inverse Gaussian Distribution*. Clarendon Press-Oxford, 1993.

[7] K. R. Baker, "On the WMC density as an Inverse Gaussian probability density," *IEEE Trans. Commun.*, vol. 44, no. 1, pp. 15–17, Jan. 1996.

[8] K. Krishnamoorthy, *Handbook of Statistical Distributions with Applications*. Chapman & Hall - CRC, 2006.

[9] R. S. Chhikara and J. L. Folks, *The Inverse Gaussian Distribution: theory, methodology, and applications*. New York: M. Dekker, 1989.

[10] A. Papoulis, *Probability, Random Variables, and Stochastic Procceses*, 3rd ed. McGraw-Hill, 1991.

[11] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 7th ed. New York: Academic, 2007.

[12] G. K. Karagiannidis and A. S. Lioumpas, "An improved approximation for the gaussian q-function," *IEEE Commun. Lett.*, vol. 11, no. 8, pp. 644–646, Aug. 2007.