On the Distribution of the Sum of Gamma-Gamma Variates and Applications in RF and Optical Wireless Communications

Nestor D. Chatzidiamantis, *Student Member, IEEE*, and George K. Karagiannidis, *Senior Member, IEEE*

***Abstract*—The Gamma-Gamma ($\Gamma\Gamma$) distribution has recently attracted the interest of the research community due to its involvement in various communication systems. In the context of RF wireless communications, $\Gamma\Gamma$ distribution accurately models the power statistics in composite shadowing/fading channels as well as in cascade multipath fading channels, while in optical wireless (OW) systems, it describes the fluctuations of the irradiance of optical signals distorted by atmospheric turbulence. Although $\Gamma\Gamma$ channel model offers analytical tractability in the analysis of single input single output (SISO) wireless systems, difficulties arise when studying multiple input multiple output (MIMO) systems, where the distribution of the sum of independent $\Gamma\Gamma$ variates is required. In this paper, we present a novel and simple closed-form approximation for the distribution of the sum of independent, but not necessarily identically distributed $\Gamma\Gamma$ variates. It is shown that the probability density function (PDF) of the $\Gamma\Gamma$ sum can be efficiently approximated either by the PDF of a single $\Gamma\Gamma$ distribution, or by a finite weighted sum of PDFs of $\Gamma\Gamma$ distributions. To reveal the importance of the proposed approximation, the performance of RF wireless systems in the presence of composite fading, as well as MIMO OW systems impaired by atmospheric turbulence, are investigated. Numerical results and simulations illustrate the accuracy of the proposed approach.**

***Index Terms*—Gamma-Gamma ($\Gamma\Gamma$) distribution, fading channels, shadowing, cascade fading, atmospheric turbulence, optical wireless, diversity reception, performance analysis.**

I. INTRODUCTION

IN communication theory, channel modeling is very crucial since it can be applied in the design and performance evaluation of various communication systems. A distribution which has recently attracted the interest within the research community due to its involvement in various communication systems, is the so-called Gamma-Gamma ($\Gamma\Gamma$) distribution. This distribution is equivalent to the squared Generalized-K (K_G) distribution [1] and can be derived from the product of two independent Gamma random variables (RVs). Moreover, for certain values of its parameters, it coincides with the K-distribution, which in the past has been widely used in a variety of applications, including the statistical characterization of the intensity of SAR images [2], as well as to model the statistics of the reverbation intensity in underwater communications [3]-[4].

Of particular interest is the application of the $\Gamma\Gamma$ distribution in optical wireless (OW) systems, where transmission of optical signals through the atmosphere is involved. In these systems, a major performance limiting factor is the *turbulence induced fading*, i.e., rapid fluctuations of the irradiance of the propagated optical signals caused by atmospheric turbulence, which can be accurately modeled using the statistics of the $\Gamma\Gamma$ distribution [5]. Furthermore, in recent years, $\Gamma\Gamma$ distribution has also been applied in the field of RF wireless communications; specifically, to model the power statistics in composite fading channels [1], [6]. Additionally, since it includes the well known Double-Rayleigh model [7], $\Gamma\Gamma$ distribution can be further employed for modeling the power statistics in cascade multipath fading channels (which occur, e.g., in keyhole or in mobile-to-mobile communication scenarios [8]).

Although $\Gamma\Gamma$ channel model is analytically tractable in the performance analysis of various single input single output (SISO) wireless communication systems [1], [9]-[12], difficulties arise when studying the performance of certain diversity schemes in multiple input multiple output (MIMO) wireless systems. Specifically, these difficulties appear when the distribution of the sum of independent $\Gamma\Gamma$ variates is required and have their origin in the fact that a straight derivation of this distribution is analytically infeasible, due to the involvement of the modified Bessel function of the second kind.

In the past, a limited number of works dealing with the distribution of the sum of independent $\Gamma\Gamma$ variates and its application in communications systems, appeared in the technical literature. In the context of mobile communications, where $\Gamma\Gamma$ distribution models the statistics of the signal-to-noise ratio (SNR) in the presence of K_G fading channel model, the maximal ratio combining (MRC) receiver has been investigated in [13] and [14]. However, in [13], the sum of independent $\Gamma\Gamma$ variates was not investigated (shadowing term was common in every diversity branch), while in [14] the expressions that were derived for the statistics of the SNR at the output of the combiner, do not hold[1]. In the context of

Paper approved by M.-S. Alouini, the Editor for Modulation and Diversity Systems of the IEEE Communications Society. Manuscript received April 17, 2009; revised March 19, 2010 and August 31, 2010.

This paper was partially presented at the IEEE Global Communications Conference (GLOBECOM'09).

The authors are with the Wireless Communications Systems Group (WCSG), Department of Electrical and Computer Engineering, Aristotle University of Thessaloniki, GR-54124 Thessaloniki, Greece (e-mail: {nestoras, geokarag}@auth.gr).

Digital Object Identifier 10.1109/TCOMM.2011.020811.090205

[1]In [14, Eq. A-2], it is assumed that the parameter $a = k - m$ is not an integer. However this comes in contradiction with the assumption that the parameters k and m in [14, Eq. A-4] are integers, which results to the incorrect expressions of [14, Eq. 11] and [14, Eq. 12].

OW communications, the statistics of the sum of $\Gamma\Gamma$ variates have been used in the performance analysis of MIMO systems operating over $\Gamma\Gamma$ turbulence model and employing equal gain combining (EGC) at the receiver. In [15], the performance of such a system was investigated over identical OW links, using an infinite power series representation for the probability density function (PDF) of the turbulence-induced fading term at the output of the receiver. Although this approach allows the derivation of simple and accurate expressions at the high SNR regime, it is not computationally attractive when the number of the transmit/receive apertures increases and/or the underlying OW links are non identically distributed.

In this paper, we address to this cumbersome statistical problem by applying a novel and simpler approach. We present novel closed-form expressions that approximate efficiently the PDF of the distribution of the sum of independent $\Gamma\Gamma$ variates based on the following well known issues: a) $\Gamma\Gamma$ distribution is derived from the product of two independently distributed Gamma RVs, and b) the distribution of the sum of Gamma variates is analytically tractable. Note that in the past several works have been published providing approximations for the statistics of the sum of RVs, where a closed form solution was impossible [16]-[19]. Our analysis encompasses the case where the variates involved in the sum are identically distributed, as well as the non identical scenario. Furthermore, in order to reveal the importance of the proposed statistical formulation, we study the performance of two different communication systems; an RF wireless system operating over the K_G fading model and employing MRC diversity scheme at the receiver and a MIMO OW system operating over strong turbulence channels and employing EGC at the receiver. For these systems, closed-form expressions that approximate significant system performance metrics, such as bit error rate (BER) and outage probability, are obtained.

The remainder of the paper is organized as follows. After a brief introduction of the $\Gamma\Gamma$ distribution in Section II, novel closed-form expressions that approximate the PDF of the sum of $\Gamma\Gamma$ variates are obtained in Section III. Moreover, in Section IV the obtained results are applied to derive closed-form expressions for the performance evaluation of an MRC receiver operating over the K_G fading model, while in Section V the same results are further applied to the performance evaluation of MIMO OW systems operating over strong turbulence channels and employing EGC at the receiver. Finally, in Section VI, useful concluding remarks are provided.

II. The $\Gamma\Gamma$ Distribution

Let $\gamma \geqslant 0$ be a three-parameter $\Gamma\Gamma$ RV derived from the square of a K_G distributed RV. Its PDF is given by [1]

$$f_\gamma(\gamma; k, m, \Omega) = \frac{2(km)^{\frac{k+m}{2}}\gamma^{\frac{k+m}{2}-1}}{\Gamma(m)\Gamma(k)\Omega^{\frac{k+m}{2}}} K_{k-m}\left[2\left(\frac{km}{\Omega}\gamma\right)^{1/2}\right] \tag{1}$$

where $k \geqslant 0$ and $m \geqslant 0$ are the distribution shaping parameters, $K_\nu(\cdot)$ is the modified Bessel function of order ν [20, 8.407/1], $\Gamma(\cdot)$ is the Gamma function [20, 8.310/1] and Ω is related with the mean as $\mathbb{E}[\gamma] = \Omega$, with $\mathbb{E}[\cdot]$ denoting expectation.

The distribution in (1) is generic, since it describes various models frequently used in communication systems, for several combinations of k and m. Hence, as $k \to \infty$, it approximates the well-known Gamma distribution (or equivalently squared Nakagami-m [21]), while for $m = 1$, it coincides with the statistics of a squared K-distributed[2] RV with PDF given by

$$f_\gamma(\gamma; k, 1, \Omega) = \frac{2k^{\frac{k+1}{2}}\gamma^{\frac{k-1}{2}}}{\Gamma(k)\Omega^{\frac{k+1}{2}}} K_{k-1}\left[2\left(\frac{k}{\Omega}\gamma\right)^{1/2}\right]. \tag{2}$$

Furthermore for the special case of $k = 1$ and $m = 1$, it reduces to the power statistics of the double Rayleigh model, frequently used in cascade multipath fading channels, with PDF given by [7]-[8] as

$$f_\gamma(\gamma; 1, 1, \Omega) = \frac{2}{\Omega} K_0\left[2\sqrt{\frac{\gamma}{\Omega}}\right]. \tag{3}$$

The n-th moment of the γ is given as [1]

$$\mathbb{E}[\gamma^n] = \xi^{-n}\frac{\Gamma(k+n)\Gamma(m+n)}{\Gamma(k)\Gamma(m)} \tag{4}$$

where $\xi = \frac{km}{\Omega}$. Moreover its cumulative density function (CDF) has been expressed using [10, Eq. 7] and [20, Eq. 9.31/5] as

$$F_\gamma(\gamma; k, m, \Omega) = \frac{1}{\Gamma(k)\Gamma(m)} G_{1,3}^{2,1}\left[\xi\gamma \left| \begin{matrix} 1 \\ k, m, 0 \end{matrix} \right.\right], \tag{5}$$

where $G[\cdot]$ is the Meijer's G function [20, Eq. 9.301].

It is important to note that the $\Gamma\Gamma$ distributed RV can be derived from the product of two independent RVs, x and y as [22]

$$\gamma = xy \tag{6}$$

when are both Gamma distributed with PDF given by

$$f_i(i; m_i, \eta_i) = \frac{i^{m_i-1}}{\eta_i^{m_i}\Gamma(m_i)} \exp\left(-\frac{i}{\eta_i}\right), \; i = x, y \tag{7}$$

and parameters $(m_x = k, \eta_x = 1/k)$ and $(m_y = m, \eta_y = \Omega/m)$ respectively[3].

III. An Efficient Approximation to the Statistics of the Sum of $\Gamma\Gamma$ Variates

Let us consider L statistically independent $\Gamma\Gamma$ variates denoted by $\{\gamma_l\}_{l=1}^L$, each having shaping parameters k_l and m_l, and mean Ω_l. Their sum, S_γ, is defined as

$$S_\gamma \triangleq \sum_{l=1}^{L}\gamma_l = \sum_{l=1}^{L} x_l y_l, \tag{8}$$

where x_l and y_l are Gamma RVs with parameters $(k_l, 1/k_l)$ and $(m_l, \Omega_l/m_l)$ respectively. Eq. (8) can be rewritten as

$$S_\gamma = \frac{\left(\sum\limits_{l=1}^{L} x_l\right)\left(\sum\limits_{l=1}^{L} y_l\right)}{L} + \frac{1}{L}\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}(x_i - x_j)(y_i - y_j). \tag{9}$$

[2]In [4] and [5], the statistics of the square of a K-distributed RV are also referred as K-distribution.

[3]It follows from symmetry that it is equivalent to consider the set of parameters $\left(m_x = k, \eta_x = \frac{\Omega}{k}\right)$ and $\left(m_y = m, \eta_y = \frac{1}{m}\right)$.

A. Identical Variates

When the variates of the sum in (8) are identically distributed (i.i.d.) (i.e. $k_l = k$, $m_l = m$, $\Omega_l = \Omega$), $\{x_l\}_{l=1}^{L}$ and $\{y_l\}_{l=1}^{L}$ are also identically distributed. Hence, according to (9), the unknown distribution of S_γ can be approximated by the distribution of the RV $\hat{S}_\gamma$, which is defined as

$$S_\gamma \approx \hat{S}_\gamma = \frac{\left(\sum_{l=1}^{L} x_l\right)\left(\sum_{l=1}^{L} y_l\right)}{L}, \tag{10}$$

with the approximation error, ε, given by

$$\varepsilon = \frac{1}{L}\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left(x_i - x_j\right)\left(y_i - y_j\right). \tag{11}$$

Equivalently, (10) can be written as the product of two RVs s_1 and s_2, i.e.

$$\hat{S}_\gamma = s_1 s_2, \tag{12}$$

where

$$s_1 = \frac{1}{L}\sum_{l=1}^{L} x_l \tag{13}$$

and

$$s_2 = \sum_{l=1}^{L} y_l. \tag{14}$$

Since the sum of i.i.d. Gamma variates remains Gamma distributed [23], it can be easily proved that s_1 and s_2 are both Gamma distributed with set of parameters $\left(Lk, \frac{1}{Lk}\right)$ and $\left(Lm, \frac{L\Omega}{Lm}\right)$ respectively. Hence, according to (6), $\hat{S}_\gamma$ will be $\Gamma\Gamma$ distributed with shaping parameters

$$k_{\hat{S}_\gamma} = Lk, \tag{15}$$

$$m_{\hat{S}_\gamma} = Lm, \tag{16}$$

and mean

$$\Omega_{\hat{S}_\gamma} = L\Omega. \tag{17}$$

The accuracy of the proposed approximation depends on the approximation error defined in (11). The exact PDF of the error is difficult to be derived, however, its first moments, which are also indicatives of its statistical behavior, can be calculated using (11). Specifically, according to the Appendix, the mean of ε is equal to 0, while its variance depends on k, m, Ω, and L, according to

$$\mathbb{E}\left[\varepsilon^2\right] = (L-1)\frac{\Omega^2}{km}. \tag{18}$$

It is obvious from the above equation that the variance of the approximation error increases for a certain combination of k, m and Ω, as the number of the RVs of the sum in (8) increases.

In order to improve the accuracy of the proposed approximation, an adjustment parameter is introduced that modifies the shaping parameters of the approximating distribution of $\hat{S}_\gamma$. Specifically, we assume that the maximum of the shaping parameters[4] $k_{\hat{S}_\gamma}$ is modified by an adjustment parameter, ε_γ, according to

$$k_{\hat{S}_\gamma} = Lk + \varepsilon_\gamma. \tag{19}$$

[4]Since $K_{-\nu}(x) = K_\nu(x)$, we assume for convenience and without loss of generality that $k_{\hat{S}_\gamma} \geqslant m_{\hat{S}_\gamma}$.

The adjustment parameter, ε_γ, is evaluated through the following rule

$$\varepsilon_\gamma = \arg\min_{\varepsilon_\gamma}\left|\mathbb{E}\left[\hat{S}_\gamma^\nu\right] - \mathbb{E}\left[S_\gamma^\nu\right]\right|, \nu = 1, ...4 \tag{20}$$

where $\mathbb{E}\left[\hat{S}_\gamma^\nu\right]$ are the moments of the distribution of $\hat{S}_\gamma$, which can be derived from (4) using the parameters $\left(k_{\hat{S}_\gamma}, m_{\hat{S}_\gamma}, \Omega_{\hat{S}_\gamma}\right)$, as provided by (19), (16) and (17) respectively; $\mathbb{E}\left[S_\gamma^\nu\right]$ are the moments of the distribution of S_γ, calculated using the multinomial expansion, according to

$$\begin{aligned}\mathbb{E}\left[S_\gamma^\nu\right] \; = \; & \sum_{\nu_1=0}^{\nu}\sum_{\nu_2=0}^{\nu_1}...\sum_{\nu_{L-1}=0}^{\nu_{L-2}}\binom{\nu}{\nu_1}\binom{\nu_1}{\nu_2}...\binom{\nu_{L-2}}{\nu_{L-1}} \\ & \times\mathbb{E}\left[\gamma_1^{\nu-\nu_1}\right]\mathbb{E}\left[\gamma_2^{\nu_1-\nu_2}\right]...\mathbb{E}\left[\gamma_L^{\nu_{L-1}}\right]\end{aligned} \tag{21}$$

and using (4) for the respective parameters (k, m, Ω).

Finding the optimum solution in (20) is a nonlinear multiple function optimization problem, which is difficult to be solved analytically, yet not impossible to derive an approximative solution numerically. After applying non-linear regression methods [24], it was found that the adjustment parameter depends on L, k and m with a function of the form of

$$\varepsilon_\gamma(L, k, m) = (L-1)\frac{-0.127 - 0.95k - 0.0058m}{1 + 0.00124k + 0.98m}. \tag{22}$$

Hence, using (22) in conjunction with (19), (16) and (17), the parameters of a single $\Gamma\Gamma$ distribution are defined, which accurately approximates the distribution of the sum of L i.i.d. $\Gamma\Gamma$ variates.

B. Non-Identical Variates

When the $\Gamma\Gamma$ variates of the sum in (8) are independent but not identically distributed (i.n.i.d.), but have one shaping parameter in common, as it happens in most practical applications, (i.e. $k_l = k$, but m_l and Ω_l are different), we still approximate the unknown PDF of the sum by the PDF of the RV $\hat{S}_\gamma$, as defined by (12).

As in the i.i.d. case, $\hat{S}_\gamma$ can be written as the product of two RVs s_1 and s_2, defined by (13) and (14) respectively. Since $k_l = k$, $\{x_l\}_{l=1}^{L}$ are i.i.d. and s_1 is Gamma distributed with parameters $\left(Lk, \frac{1}{Lk}\right)$. However, the derivation of the distribution of s_2 is not straightforward, since $\{y_l\}_{l=1}^{L}$ are not identically distributed. In order to derive the PDF of s_2, the exact closed-form expressions for the sum of non-identical Gamma variates, presented in [25], are used. According to this approach, the PDF of s_2 can be written as a nested finite weighted sum of Gamma PDFs,

$$f_{s_2}(z) = \sum_{i=1}^{L}\sum_{j=1}^{m_i} w_L\left(i, j, \{m_l\}_{l=1}^{L}, \{\Omega_l\}_{l=1}^{L}\right) f_y\left(z; j, \frac{\Omega_i}{m_i}\right), \tag{23}$$

where y is a Gamma distributed RV with PDF given by (7) and the weights can be easily and quickly evaluated using the recursive formula of [25, Eq. (8)] for the parameters $\{m_l\}_{l=1}^{L}$

and $\{\Omega_l\}_{l=1}^{L}$, according to

$$w_L\left(i, m_i - t, \{m_l\}_{l=1}^{L}, \{\Omega_l\}_{l=1}^{L}\right) = \frac{1}{t}\sum_{\substack{q=1\\q\neq i}}^{L}\sum_{j=1}^{t}\frac{m_q}{\frac{\Omega_i^j}{m_i^j}}\left(\frac{m_i}{\Omega_i}-\frac{m_q}{\Omega_q}\right)^{-j} \times w_L\left(i, m_i - t + j, \{m_l\}_{l=1}^{L}, \{\Omega_l\}_{l=1}^{L}\right) \quad (24)$$

where $t = 1, ..., m_i - 1$ and with

$$w_L\left(i, m_i, \{m_l\}_{l=1}^{L}, \{\Omega_l\}_{l=1}^{L}\right) = \frac{\frac{\Omega_i^{m_i}}{m_i^{m_i}}}{\prod_{h=1}^{L}\frac{\Omega_h^{m_h}}{m_h^{m_h}}}\prod_{\substack{j=1\\j\neq i}}^{L}\left(\frac{m_j}{\Omega_j}-\frac{m_i}{\Omega_i}\right)^{-m_j}. \quad (25)$$

The PDF of the product of s_1 and s_2 is evaluated as

$$f_{\hat{S}_\gamma}(z) = \int_0^\infty \frac{1}{x} f_{s_1}\left(x; Lk, \frac{1}{Lk}\right) f_{s_2}\left(\frac{z}{x}\right) dx. \quad (26)$$

Using (23) and [20, Eq. 3.471/9], Eq. (26) yields as

$$f_{\hat{S}_\gamma}(z) = \sum_{i=1}^{L}\sum_{j=1}^{m_i} w_L\left(i, j, \{m_l\}_{l=1}^{L}, \{\Omega_l\}_{l=1}^{L}\right) \times f_\gamma\left(z; Lk, j, \frac{j\Omega_i}{m_i}\right), \quad (27)$$

where γ is a $\Gamma\Gamma$ distributed RV with PDF defined by (1). Hence, an efficient approximation to the PDF of the sum of L non-identical $\Gamma\Gamma$ variates, when one of the shaping parameters remains the same for all variates[5], can be a nested finite weighted sum of $\Gamma\Gamma$ PDFs.

C. KS Goodness-of-fit tests

Next, we investigate the validity of the proposed approximations using statistical tools and arguments. Specifically, we employ Kolmogorov-Smirnov (KS) goodness-of-fit statistical tests [26, pp. 272-273] that measure the maximum value of the absolute difference between the empirical CDF of the RV S_γ, $F_{S_\gamma}(\cdot)$, and the analytical CDF of the approximating RV $\hat{S}_\gamma$, $F_{\hat{S}_\gamma}(\cdot)$. Hence, the KS test statistic is defined as

$$T \triangleq \max\left|F_{S_\gamma}(z) - F_{\hat{S}_\gamma}(z)\right|. \quad (28)$$

In the i.i.d. case, $\hat{S}_\gamma$ is a single $\Gamma\Gamma$ variate with set of parameters $\left(k_{\hat{S}_\gamma}, m_{\hat{S}_\gamma}, \Omega_{\hat{S}_\gamma}\right)$ obtained by (19), (16) and (17) respectively; thus, $F_{\hat{S}_\gamma}(z)$ can be derived, according to (5), as

$$F_{\hat{S}_\gamma}(z) = F_\gamma\left(z; k_{\hat{S}_\gamma}, m_{\hat{S}_\gamma}, \Omega_{\hat{S}_\gamma}\right), \quad (29)$$

[5]Note that due to symmetry, the same approximation also holds when $m_i = m$ and k_i are different, by interchanging k_i and m_i in (27), i.e. $m_i = k_i$ and $k = m$.

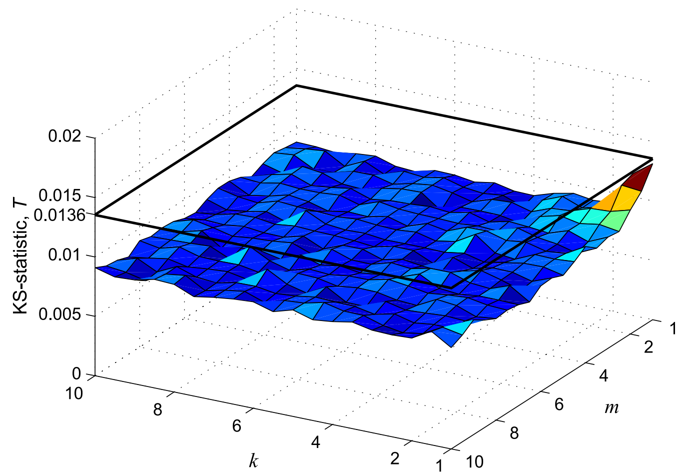


Fig. 1. Hypothesis testing distribution using the KS goodness-of-fit test for the distribution of $\hat{S}_\gamma$ to approximate the distribution of S_γ with 5% significance level, when $L = 2$.

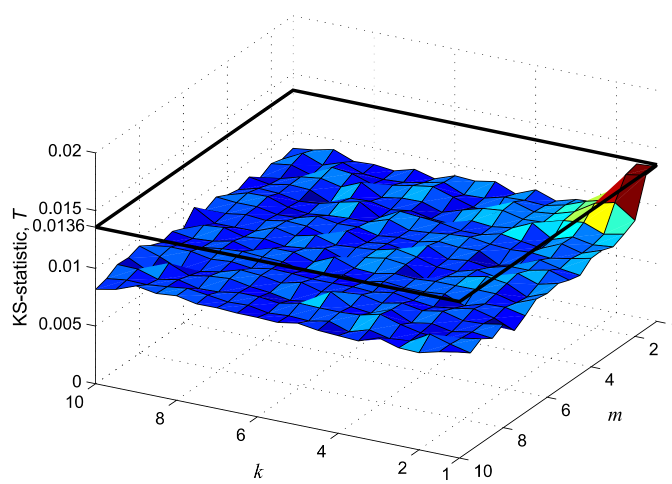


Fig. 2. Hypothesis testing distribution using the KS goodness-of-fit test for the distribution of $\hat{S}_\gamma$ to approximate the distribution of S_γ with 5% significance level, when $L = 3$.

while in the i.n.i.d. case, $F_{\hat{S}_\gamma}(z)$ can be derived, according to (27), as

$$F_{\hat{S}_\gamma}(z) = \sum_{i=1}^{L}\sum_{j=1}^{m_i} w_L\left(i, j, \{m_l\}_{l=1}^{L}, \{\Omega_l\}_{l=1}^{L}\right) \times F_\gamma\left(\gamma; Lk, j, \frac{j\Omega_i}{m_i}\right). \quad (30)$$

Definition 1: We define $\boldsymbol{H}_0$ as the null hypothesis under which the observed data of S_γ belong to the CDF of the approximating distribution $F_{\hat{S}_\gamma}(\cdot)$.

To test $\boldsymbol{H}_0$, the KS goodness-of-fit test compares the test statistic T to a critical level $T_{\max}$ for a given significance level α. Any hypothesis for which $T > T_{\max}$, is rejected with significance $1-\alpha$, while any hypothesis for which $T < T_{\max}$ is accepted with the same level of significance.

Figs. 1 and 2 depict the KS test statistic for different combinations of the parameters k ,m and L, when the i.i.d. sum of variates is considered. Since the parameter Ω does not influence the shape of the approximating distribution in

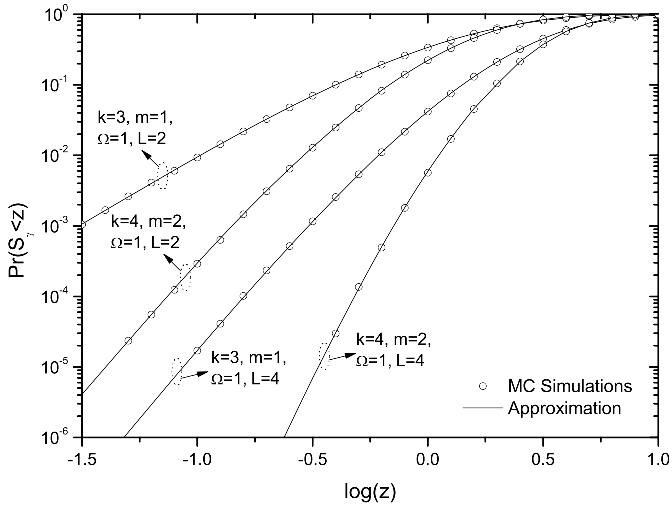


Fig. 3. The CDF plots of the RV S_γ and the approximating RV $\hat{S}_\gamma$ for different combinations of parameters and i.i.d. sum of variates.

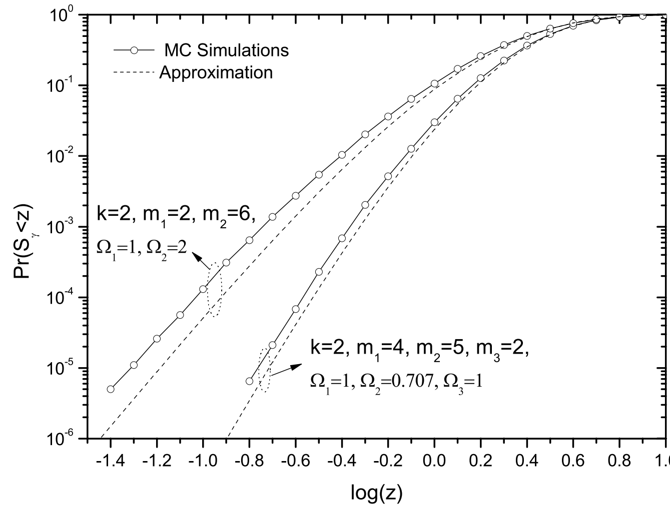


Fig. 4. The CDF plots of the RV S_γ and the approximating RV $\hat{S}_\gamma$ for different combinations of parameters.

the i.i.d. case, without loss of generality, it is assumed that $\Omega = 1$. The presented results have been obtained by averaging the results of 60 simulation runs, each for at least 10^4 samples of the RV S_γ. The critical value $T_{\max}$, used for comparison reasons, is equal to $T_{\max} = 0.0136$ which corresponds to a significance level of $\alpha = 5\%$ [26, Eq. (9-73)]. As it is clearly illustrated in the figures, although the approximation looses its high accuracy in the region $k \leq 1$ and $m \leq 1$ (T comes closer to the critical level $T_{\max}$), the hypothesis $\boldsymbol{H}_0$ is accepted with 95% significance in all the range of the parameters examined, even when L increases. Hence, in the i.i.d. case, the distribution of $\hat{S}_\gamma$ can be considered as a highly efficient approximation of the distribution of S_γ. The same conclusion can also be reached by observing the CDF plots of Fig. 3 for some combinations of k, m and L.

When the variates of the sum in (8) are i.n.i.d., the hypothesis $\boldsymbol{H}_0$ is not always accepted with 95% significance and the accuracy of the proposed approximation depends on the combinations of the parameters k, $\{m_l\}_{l=1}^{L}$ and $\{\Omega_l\}_{l=1}^{L}$. For instance, when the following typical values are considered, $k = 2$, $m_1 = 4$, $m_2 = 5$, $m_3 = 3$, $\Omega_1 = 1$, $\Omega_2 = \sqrt{2}$ and $\Omega_3 = 1$, the hypothesis $\boldsymbol{H}_0$ is accepted with 95% significance ($T = 0.0125$), while for the parameters $k = 2$, $m_1 = 2$, $m_2 = 6$, $\Omega_1 = 1$ and $\Omega_2 = 2$, the hypothesis $\boldsymbol{H}_0$ is rejected with the same level of significance ($T = 0.0163$). Nevertheless, as it can be observed by the Fig. 4, in all cases the CDF of the approximating distribution acts as a lower bound.

IV. Application in RF Wireless Systems

A. System Model

Let us consider a diversity receiver with L branches operating over the composite K_G fading channel [14]. The equivalent complex baseband received signal at the lth ($l = 1, 2, ..., L$) branch is given by

$$z_l = sh_l + n_l \tag{31}$$

where s is the transmitted complex symbol with energy $\mathbb{E}\left[|s^2|\right]$ and $|\cdot|$ denoting absolute value, h_l is the channel's complex gain in the path between the transmitter and the lth branch, and n_l is the complex Additive White Gaussian Noise (AWGN), having single sided power spectral density N_o and assumed to be identical in all branches.

Since operation over K_G fading channel model is considered, the square of the fading envelope, $R_l^2 = |h_l|^2$, is statistically described by the PDF of Eq. (1) with shaping parameters k_l and m_l, and mean $\mathbb{E}\left[R_l^2\right] = \Omega_l$. It follows that the instantaneous SNR of the lth receiving branch, which is defined as

$$\gamma_l = \frac{R_l^2 E_s}{N_o}, \tag{32}$$

is also $\Gamma\Gamma$ distributed with PDF given by (1), shaping parameters equal to k_l and m_l, and mean equal to the average input SNR of the branch defined by

$$\overline{\gamma}_l = \frac{\Omega_l E_s}{N_o}. \tag{33}$$

Furthermore, maximum ratio combining (MRC) technique is applied at the receiver and hence the total SNR per symbol at the output of the receiver is

$$\gamma_T = \sum_{l=1}^{L} \gamma_l = \frac{E_s}{N_o} \sum_{l=1}^{L} R_l^2. \tag{34}$$

In the analysis that follows, it is assumed that both shadowing and multipath fading effects are independent among the diversity branches, i.e. macrodiversity is studied. As a consequence, the diversity branches are considered independent. Moreover, since shadowing occurs in large geographical areas, it is further assumed that the parameter that statistically describes the channel's shadowing effects, k_l, remains constant among the diversity branches, i.e., $k_l = k$ [14]. Note that the assumption of the uncorrelated diversity branches can be further applied in the scenario of diversity reception in cascade multipath fading channels, when a rich scattering radio environment is considered (see [8] for examples).

B. Error Analysis

The average BER of the under consideration RF system can be evaluated directly by averaging the conditional BER, $P_e(\gamma)$, which depends from the type of modulation, over the PDF of γ_T, $f_{\gamma_T}(\gamma)$, i.e.,

$$\overline{P}_{be} = \int_0^\infty P_e(\gamma) f_{\gamma_T}(\gamma)\, d\gamma. \tag{35}$$

1) Independent and Identically Distributed Diversity Branches: When the received signals at the diversity branches are independent and identically distributed (i.i.d.), i.e. $m_l = m$ and $\overline{\gamma}_l = \overline{\gamma}$, the approximation for the PDF of the sum of i.i.d. $\Gamma\Gamma$ variates can be used in order to evaluate the error performance of the under consideration system. Specifically, the PDF of γ_T can be approximated by the PDF of a single $\Gamma\Gamma$ variate with parameters obtained by (19), (16) and (17), i.e.

$$f_{\gamma_T}(\gamma) \approx f_\gamma(\gamma; k_T, m_T, \overline{\gamma}_T) \tag{36}$$

where $k_T = Lk + \varepsilon_{\gamma_1}$, $m_T = Lm$, $\overline{\gamma}_T = L\overline{\gamma}$ and ε_{γ_1} is calculated using (22) for the set of parameters of (L, k, m). By using the corresponding BER expressions of a Single Input Single Output (SISO) system with parameters $(k_T, m_T, \overline{\gamma}_T)$, analytical expressions that approximate the average BER of the diversity system can be derived. Hence, using [1, Eq. (8)], the average BER for BPSK modulation can be approximated by

$$\overline{P}_{be} \approx \frac{\xi_T^{\frac{k_T+m_T}{2}}}{2\sqrt{\pi}\Gamma(k_T)\Gamma(m_T)} G_{2,3}^{2,2}\left[\xi_T \left| \begin{matrix} \frac{1-\beta_T}{2}, -\frac{\beta_T}{2} \\ \frac{a_T}{2}, -\frac{a_T}{2}, -\frac{\beta_T+1}{2} \end{matrix} \right.\right] \tag{37}$$

where $\xi_T = \frac{k_T m_T}{\overline{\gamma}_T}$, $\beta_T = k_T + m_T - 1$ and $a_T = k_T - m_T$. In a similar manner, using [1, Eq. (9)], the BER performance of the DBPSK diversity system is evaluated as

$$\overline{P}_{be} \approx \frac{1}{2}\xi_T^{\frac{\beta_T}{2}} \exp\left(\frac{\xi_T}{2}\right) W_{-\frac{\beta_T}{2}, \frac{a_T}{2}}(\xi_T) \tag{38}$$

where $W_{\lambda,\mu}(\cdot)$ is the Whittaker function [20, Eq. 9.220].

2) Independent, but not Necessarily Identically Distributed Diversity Branches: When the received signals at the diversity branches are independent, but not necessarily identically distributed (i.n.i.d.), i.e., m_l and $\overline{\gamma}_l$ are different among the diversity branches, the approximation for the PDF of the sum of i.n.i.d. $\Gamma\Gamma$ variates can be used in order to approximate the average BER of the under consideration diversity system. According to (27), the PDF of γ_T can be approximated by a nested finite weighted sum of $\Gamma\Gamma$ PDFs, i.e.

$$f_{\gamma_T}(\gamma) \approx \sum_{i=1}^{L}\sum_{j=1}^{m_i} \Xi(i,j)\, f_\gamma\left(\gamma; Lk, j, \frac{j\overline{\gamma}_i}{m_i}\right), \tag{39}$$

where $\Xi(i,j) = w_L\left(i, j, \{m_l\}_{l=1}^{L}, \{\overline{\gamma}_l\}_{l=1}^{L}\right)$. Hence, according to (35), the error performance can be evaluated using the corresponding BER expressions of a SISO system with parameters $\left(Lk, j, \frac{j\overline{\gamma}_i}{m_i}\right)$. Specifically, using [1, Eq. (8)], an analytical expression that approximates the average BER of the BPSK diversity system yields as

$$\overline{P}_{be} \approx \sum_{i=1}^{L}\sum_{j=1}^{m_i} \frac{\Xi(i,j)\,\xi_i^{\frac{Lk+j}{2}}}{2\sqrt{\pi}\Gamma(Lk)\Gamma(j)} G_{2,3}^{2,2}\left[\xi_i \left| \begin{matrix} \frac{1-\beta_j}{2}, -\frac{\beta_j}{2} \\ a_j, -a_j, -\frac{\beta_j+1}{2} \end{matrix} \right.\right] \tag{40}$$

where $\xi_i = \frac{Lkm_i}{\overline{\gamma}_i}$, $a_j = \frac{Lk-j}{2}$ and $\beta_j = Lk + j - 1$, while using [1, Eq. (9)], the average BER of the DBPSK diversity system is derived as

$$\overline{P}_{be} \approx \sum_{i=1}^{L}\sum_{j=1}^{m_i} \frac{\Xi(i,j)}{2}\xi_i^{\frac{Lk+j-1}{2}} \exp\left(\frac{\xi_i}{2}\right) W_{-\frac{\beta_j}{2}, \frac{a_j}{2}}(\xi_i). \tag{41}$$

C. Outage Probability

Outage probability is defined as the probability that the output SNR of the under consideration diversity system falls below a specified threshold γ_{th}, which represents a protection value of the SNR above which the quality of the channel is satisfactory. Hence, outage probability can be evaluated by

$$P_{out} = \Pr(\gamma_T \le \gamma_{th}) = \int_0^{\gamma_{th}} f_{\gamma_T}(\gamma)\, d\gamma. \tag{42}$$

1) Independent and Identically Distributed Diversity Branches: When the received signals at the diversity branches are i.i.d., i.e. $m_l = m$ and $\overline{\gamma}_l = \overline{\gamma}$, the outage probability of the diversity system can be approximated by the outage probability of a SISO system with parameters $(k_T, m_T, \overline{\gamma}_T)$. Using (5), a closed form solution yields as

$$P_{out} \approx \frac{1}{\Gamma(k_T)\Gamma(m_T)} G_{1,3}^{2,1}\left[\xi_T\gamma_{th} \left| \begin{matrix} 1 \\ k_T, m_T, 0 \end{matrix} \right.\right] \tag{43}$$

where $\xi_T = \frac{k_T m_T}{\overline{\gamma}_T}$.

2) Independent but not Necessarily Identically Distributed Diversity Branches: When the received signals at the diversity branches are i.n.i.d., i.e. m_l and $\overline{\gamma}_l$ are different among the diversity branches, the outage probability can be approximated by a nested finite weighted sum of of outage probabilities of SISO systems, each having the parameters $\left(Lk, j, \frac{j\overline{\gamma}_i}{m_i}\right)$. Hence, by using (5), an approximative expression for the outage probability is obtained as

$$P_{out} \approx \sum_{i=1}^{L}\sum_{j=1}^{m_i} \Xi(i,j) \frac{1}{\Gamma(Lk)\Gamma(j)} G_{1,3}^{2,1}\left[\xi_i\gamma_{th} \left| \begin{matrix} 1 \\ Lk, j, 0 \end{matrix} \right.\right] \tag{44}$$

where $\xi_i = \frac{Lkm_i}{\overline{\gamma}_i}$ and the weights $\Xi(i,j)$ are defined earlier.

D. Numerical Results and Discussion

Figs. 5 and 6 illustrate the average BER and outage probability of MRC receivers operating over the K_G fading model, when the diversity branches have the same shaping parameters and average branch input SNR, i.e., $k_l = k$, $m_l = m$ and $\overline{\gamma}_l = \overline{\gamma}$. Approximative analytical results, using (37) and (38) for the BER evaluation and (43) for the outage probability evaluation, are plotted in comparison with Monte-Carlo (MC) simulation results, for an arbitrary number of diversity branches and assuming a certain combination of shaping parameters ($k = 2$ and $m = 5$). It is observed that there is an excellent match between simulation and

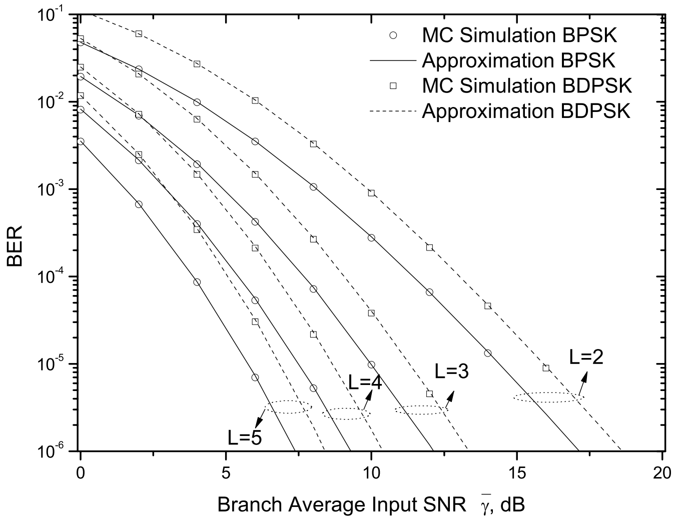


Fig. 5. Comparison of approximate average BER and MC simulation results of MRC receivers with i.i.d.diversity branches ($k = 2$ and $m = 5$).

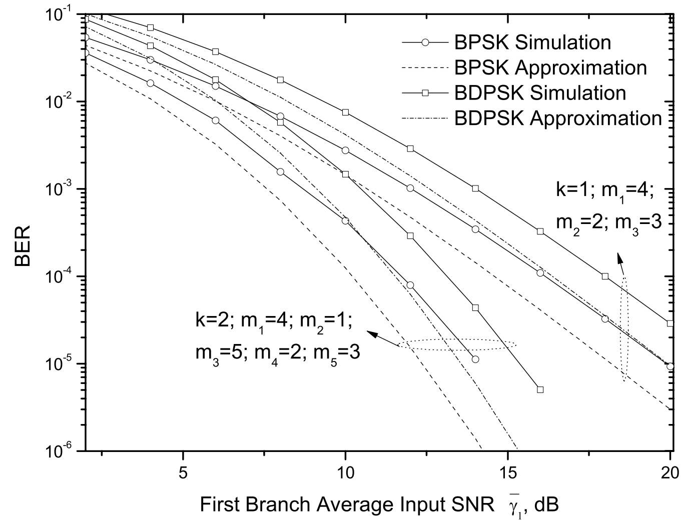


Fig. 7. Comparison of approximate average BER and MC simulation results of MRC receivers with i.n.i.d. diversity branches ($\delta = 1$).

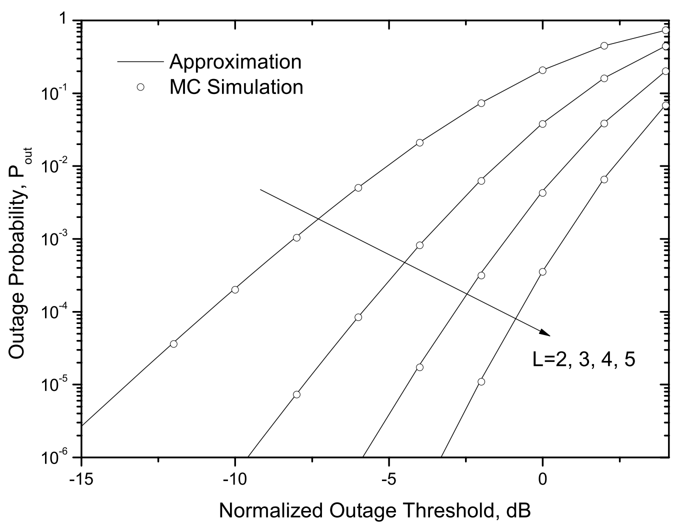


Fig. 6. Comparison of approximate Outage probability and MC simulation results of MRC receivers with i.i.d. diversity branches ($k = 2$ and $m = 5$).

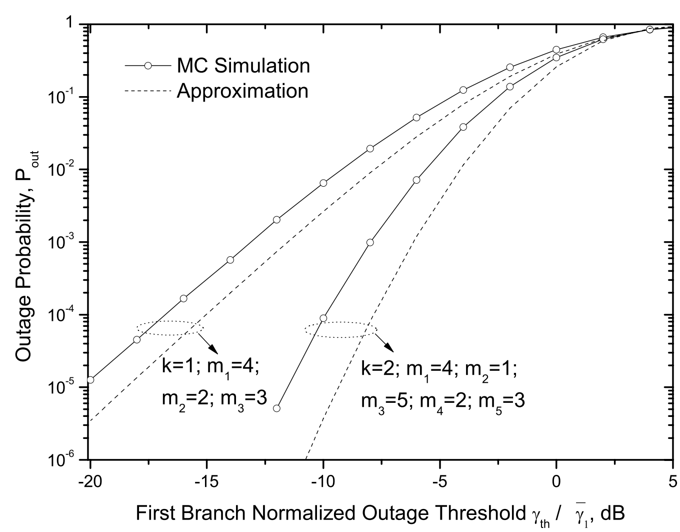


Fig. 8. Comparison of approximate Outage probability and MC simulation results of MRC receivers with i.n.i.d. diversity branches ($\delta = 1$).

approximative results for every input SNR and normalized outage, $\gamma_{th}/\overline{\gamma}$, in both performance metrics. It is also clearly depicted that the approximative analytical expressions remain accurate, even when the number of the diversity branches increases.

In Figs. 7 and 8, the BER and outage probability performance metrics of MRC receivers are depicted, when the diversity branches are i.n.i.d., i.e., m_l and $\overline{\gamma}_l$ are different among the diversity branches. Specifically, it is assumed that the average input SNR of the lth branch is given by $\overline{\gamma}_l = \overline{\gamma}_1 \exp\left[-\delta\left(l-1\right)\right]$, where $\bar{\gamma}_1$ is the average input SNR of the first branch and δ is a decaying factor. Using (40) for BPSK and (41) for DBPSK modulation, Fig. 7 presents the approximative analytical results for the average BER, as a function of the first brach average input SNR, for several combinations of shaping parameters k and m_l ($\delta = 1$). It is obvious from the figures that the approximative analytical expressions for the average BER are close to simulation results (their difference is not greater than 3 dB at target BERs equal to 10^{-5}). Moreover it is evident that the proposed approximation acts as a lower bound for high values of SNR and the smaller the number of diversity branches are, the more accurate is the bound. Similar behavior has been also observed for the outage probability, which is depicted in Fig. 8 as a function of the first branch normalized outage threshold, $\gamma_{th}/\overline{\gamma}_1$, for the same combinations of k, m_l and δ. The difference between analytical results, derived from (44), and simulation results also lies within 3dB in all cases examined, and the proposed approximation acts as a tight lower bound.

V. Application in Optical Wireless Systems

A. System Model

Consider a Multiple Input Multiple Output (MIMO) optical wireless (OW) system where the information signal is transmitted via M apertures and received by N apertures over strong atmospheric turbulence conditions. For the OW system under consideration, it is assumed that the information bits are modulated using On-Off keying (OOK) and transmitted through the M apertures using repetition coding [27]. Moreover, a large field of view is considered for each receiver, indicating that multiple transmitters are simultaneously observed by each receiver. This actually leads to the collection of larger amount of background radiation which justifies the use of the AWGN model as a good approximation of the Poisson photon counting detection model [28]. Hence, the received signal at the qth receive aperture is given by

$$r_q = x\eta \sum_{p=1}^{M} I_{pq} + \upsilon_q,\ q = 1, \ldots N \tag{45}$$

where $x \in \{0,1\}$ represents the information bits, η is the optical-to-electrical conversion coefficient and υ_q is the AWGN with zero mean and variance $\sigma_\upsilon^2 = N_o/2$.

The term I_{pq} denotes the fading coefficient that models the atmospheric turbulence through the optical channel between the pth transmit and the qth receive aperture. Since operation under strong atmospheric turbulence conditions is assumed, according to [9], the parameter which represents the effective number of small scale scatterers can be considered equal to 1. Hence, the optical channel in the p-qth transmit-receive pair can be statistically described by a $\Gamma\Gamma$ distribution with parameters $k = 1$, $m = a_{pq}$ and $\Omega = \mathbb{E}[I_{pq}]$ [9], where a_{pq} is related to the effective number of large scale scatterers. Furthermore, it is assumed that the statistics of the fading coefficients of the underlying channels are statistically independent; an assumption which is realistic by placing the transmitter and the receiver apertures just a few centimeters apart [29].

At the receiver side, the received optical signals from the N apertures are combined using equal gain combining (EGC), which is an efficient combining scheme in OW systems [28]-[29]. Hence, the output of the receiver is

$$r = \sum_{q=1}^{N} r_q = \frac{x\eta}{MN} \sum_{q=1}^{N}\sum_{p=1}^{M} I_{pq} + \upsilon. \tag{46}$$

Note that a scaling factor of MN appears in (46). The factor M is included in order to ensure that the total transmit power is the same with that of a system with no transmit diversity, while the factor N ensures that the sum of the N receive aperture areas is the same with the aperture area of a system with no receive diversity.

The received electrical SNR of the OW link between the p transmit and q receive aperture, can be defined as [30]

$$h_{pq} = \frac{\eta^2 I_{pq}^2}{N_o}, \tag{47}$$

while its average as $\mu_{pq} = \frac{\eta^2 E[I_{pq}]^2}{N_o}$. According to the above definitions, the electrical SNR of the combined signal at the output of the receiver, becomes

$$h_T = \frac{\eta^2 (I_T)^2}{M^2 N^2 N_o}, \tag{48}$$

where $I_T = \sum_{q=1}^{N}\sum_{p=1}^{M} I_{pq}$.

B. Error Analysis

The BER probability of the MIMO OW system under consideration, assuming perfect Channel State Information (CSI), is given by [28] as

$$P_e = \int_{\mathbf{I}} f_{\mathbf{I}}(\mathbf{I})\, Q\left(\frac{\eta}{2MN\sigma_\upsilon}\sum_{q=1}^{N}\sum_{p=1}^{M} I_{pq}\right) d\mathbf{I} \tag{49}$$

where $f_{\mathbf{I}}(\mathbf{I})$ is the joint PDF of the vector $\mathbf{I} = (I_{11}, I_{12}, \ldots I_{MN})$ of length MN. Furthermore, $Q(\cdot)$ is the Gaussian-Q function defined as $Q(y) = \left(1/\sqrt{2\pi}\right)\int_y^\infty \exp\left(-t^2/2\right) dt$ and related to $\operatorname{erfc}(\cdot)$ by $\operatorname{erfc}(x) = 2Q\left(\sqrt{2}x\right)$. Equivalently, Eq. (49) can be evaluated as

$$P_e = \frac{1}{2}\int_0^\infty f_{I_T}(I)\operatorname{erfc}\left(\frac{\eta}{2\sqrt{2}NM\sigma_\upsilon} I\right) dI \tag{50}$$

where $f_{I_T}(I)$ is the PDF of I_T.

1) Independent and Identically Distributed OW Links: When the turbulence induced fading coefficients of the underlying optical links of the MIMO system are independent and identically distributed, i.e. $a_{pq} = a$ and $\mathbf{E}[I_{pq}] = I_o$, the PDF of I_T can be approximated by the PDF of a single $\Gamma\Gamma$ variate, i.e.

$$f_{I_T}(I) \approx f_\gamma(I; k_T, m_T, \Omega_T) \tag{51}$$

where $k_T = MNa + \varepsilon_{\gamma_2}$, $m_T = MN$, $\Omega_T = MNI_o$ and ε_{γ_2} is calculated from (22) for the parameters $(MN, a, 1)$. Hence, the BER probability of (50) is approximated by

$$P_e \approx \frac{1}{2}\int_0^\infty f_\gamma(I; k_T, m_T, \Omega_T)\operatorname{erfc}\left(\frac{\eta}{2\sqrt{2}NM\sigma_\upsilon} I\right) dI \tag{52}$$

The integral of (52) can be solved using Meijer's G-functions and their properties. Hence, by substituting the PDF of the $\Gamma\Gamma$ distribution according to (1), expressing the $K_\nu(\cdot)$ and the $\operatorname{erfc}(\cdot)$ integrands in terms of Meijer's G-function according to [31, Eq. (8.4.23.1)] and [31, Eq. (8.4.14.2)] respectively, and using [31, Eq. (2.24.1.1)], the BER is expressed as

$$P_e \approx \frac{2^{k_T+m_T-3}}{\sqrt{\pi^3}\Gamma(k_T)\Gamma(m_T)} \times G_{5,2}^{2,4}\left[\left(\frac{2}{k_T m_T}\right)^2 \mu \,\middle|\, \begin{matrix} \frac{1-k_T}{2}, \frac{2-k_T}{2}, \frac{1-m_T}{2}, \frac{2-m_T}{2}, 1 \\ 0, \frac{1}{2} \end{matrix}\right] \tag{53}$$

where μ denotes the average electrical SNR of each OW link.

2) Independent and Not Identically Distributed OW Links: When the turbulence induced fading coefficients of the underlying optical links of the MIMO OW system are independent, but not identically distributed, the PDF of I_T can be approximated by a nested finite weighted sum of $\Gamma\Gamma$ PDFs, according to (27), i.e.

$$f_{I_T}(I) \approx \sum_{i=1}^{L}\sum_{j=1}^{m_i} \Xi(i,j)\, f_\gamma\left(I; L, j, \frac{j\Omega_i}{m_i}\right) \tag{54}$$

where $L = MN$ is the number of the underlying OW links, $m_l = a_{pq}$, $k_l = 1$ and $\Omega_l = \mathrm{E}[I_{pq}]$, when $p = 1, \ldots M$, $q = 1, \ldots N$ and $l = 1, \ldots MN$. Furthermore, the weights

$\Xi(i,j) = w_L\left(i,j,\{m_l\}_{l=1}^{L},\{\Omega_l\}_{l=1}^{L}\right)$ are evaluated using (24) and (25).

Using (54), the BER probability of the MIMO OW system is approximated by

$$P_e \approx \sum_{i=1}^{L}\sum_{j=1}^{m_i}\frac{\Xi(i,j)}{2}\int_0^{\infty} f_\gamma\left(I;L,j,\frac{j\Omega_i}{m_i}\right) \times \operatorname{erfc}\left(\frac{\eta}{2\sqrt{2}NM\sigma_v}I\right)dI \quad (55)$$

The integral in the above equation can be evaluated by expressing its integrands in terms of Meijer's G-function, as in the i.i.d. case. Hence, the probability of error is given by

$$P_e \approx \sum_{i=1}^{L}\sum_{j=1}^{m_i}\frac{2^{L+j-3}\Xi(i,j)}{\sqrt{\pi^3}\Gamma(L)\Gamma(j)} \times G_{5,2}^{2,4}\left[\left(\frac{2}{L^2 m_i}\right)^2\mu_i \,\middle|\, \begin{matrix}\frac{1-L}{2},\frac{2-L}{2},\frac{1-j}{2},\frac{2-j}{2},1\\ 0,\frac{1}{2}\end{matrix}\right] \quad (56)$$

where μ_i is the average electrical SNR of the ith OW link.

C. Outage Probability

Similarly to the wireless RF systems, outage probability in OW systems is defined as the probability that the SNR of the combined signal at the output of the receiver falls below a specified threshold h_{th}, i.e.,

$$P_{out} = \Pr(h_T \le h_{th}) = \Pr(I_T \le I_{th}) \quad (57)$$

where $I_{th} = \frac{NM}{\eta}\sqrt{h_{th}N_o}$ is the normalized threshold. This metric is considered as an important parameter for OW links to be operated as a part of a data network and, further, is critical in the design of both transport and network layer.

1) Independent and Identically Distributed OW Links: When the underlying channels of the MIMO OW system are independent and identically distributed, the outage probability of the under consideration system is equivalent to the outage probability of a SISO system operating over the ΓΓ turbulence model with parameters (k_T, m_T, Ω_T). Hence, by using (5), an approximative closed-form expression for the outage probability which is similar to (43), is obtained.

2) Independent and Not Identically Distributed OW Links: When the underlying channels of the MIMO OW system are independent, but not identically distributed, its outage probability can be approximated, according to (54), by a finite nested weighted sum of outage probabilities of SISO OW links operating over the ΓΓ turbulence model with parameters $\left(L, j, \frac{j\Omega_i}{m_i}\right)$. Hence, an approximative closed-form expression for the outage probability, in analogy to (44), can be derived.

D. Numerical Results

In Figs. 9 and 10, the BER and outage probability of MIMO OW systems operating over identically distributed strong turbulence channels with parameters $a = 4$ or $a = 10$ and $I_o = 1$, are depicted. Analytical results are illustrated in comparison with MC simulations. It is observed that there is an excellent match between the approximation and the simulations in every SNR regime for both performance metrics. It is also clearly

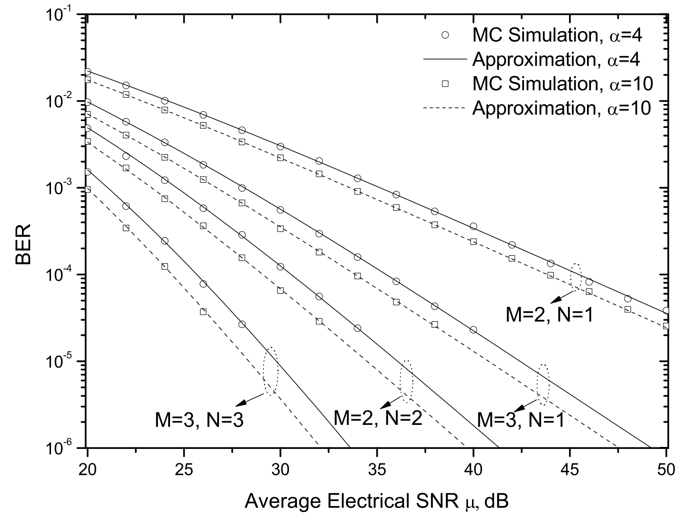


Fig. 9. Comparison of approximate average BER and MC simulation results for various MIMO OW systems over i.i.d. links.

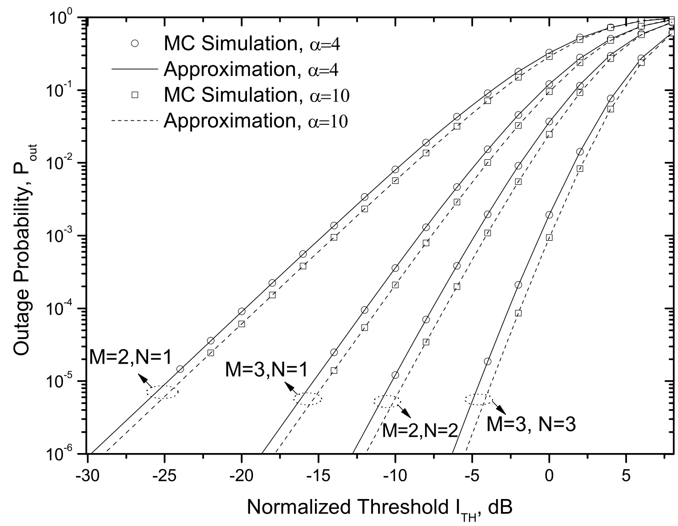


Fig. 10. Comparison of approximate outage probability and MC simulation results for various MIMO OW systems over i.i.d. links.

depicted that the derived approximative expressions are accurate for every MIMO deployment investigated, irrespective the number of transmit and/or receive apertures.

Figs. 11 and 12 depict the BER and outage probability of various MIMO deployments of transmit and receive apertures over i.n.i.d. strong turbulence channels, i.e. the underlying OW links have different turbulence parameters and different average electrical SNRs. As it is clearly illustrated, the approximative analytical results for both performance metrics respectively, are very close to the MC simulation results. Specifically, for the MIMO deployments investigated and for practical values of average BER and outage probability, the difference between analytical and simulation results is not greater than 2 dB. Moreover, it is observed that the proposed approximation acts as a lower bound, which becomes less accurate as the number of the underlying i.n.i.d. OW links increases. However, taking into consideration that the BER performance metric is difficult or even impossible to be evaluated with numerical techniques as the number of the OW

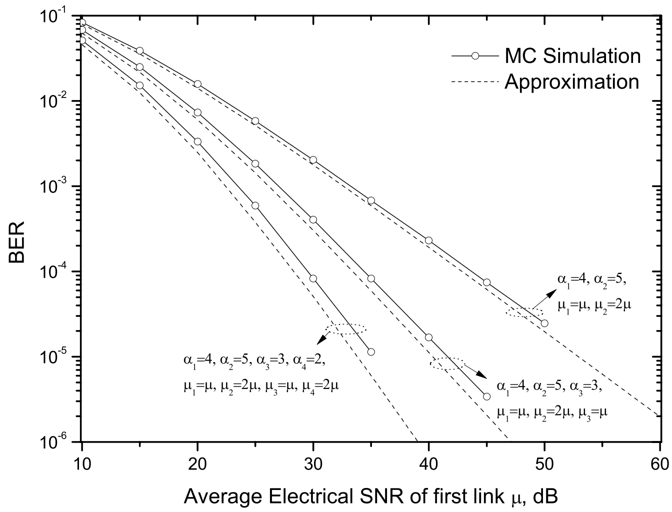


Fig. 11. Comparison of approximate BER performance and MC simulation results of MIMO systems with $M = 2$ and $N = 1$, $M = 3$ and $N = 1$, and $M = 2$ and $N = 2$ transmit and receive apertures.

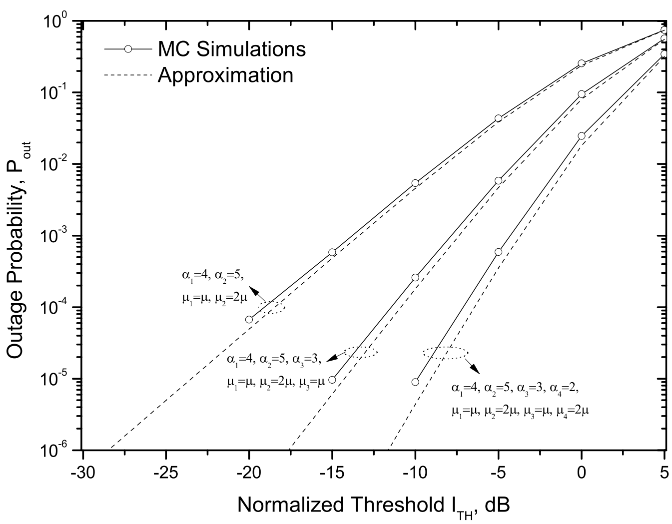


Fig. 12. Comparison of approximate outage probability and MC simulation results of MIMO systems with $M = 2$ and $N = 1$, $M = 3$ and $N = 1$, and $M = 2$ and $N = 2$ transmit and receive apertures.

links increases [30], the derived closed-form expressions can be considered as reliable alternatives to time consuming MC simulations.

VI. Conclusions

We examined the statistics of the sum of independent and not necessarily identical $\Gamma\Gamma$ RVs. Novel closed-form expressions were derived that approximated its PDF either with the PDF of a single $\Gamma\Gamma$ distribution, when all the variates of the sum were identically distributed, or with a finite weighted sum of PDFs of $\Gamma\Gamma$ distributions, when the variates of the sum were non identically distributed. Based on the obtained statistical formulas, the performance of MRC diversity receivers operating over the K_G fading channel, as well as MIMO OW systems operating over strong turbulence channels and employing EGC at the receiver, was investigated and major performance metrics were analytically evaluated. The comparison between approximative analytical results and simulations demonstrated that the proposed approximation is accurate when the diversity branches or underlying OW links are identical, while it serves as a tight lower bound for non identical diversity branches or OW links.

Appendix

Since both $\{x_l\}_{l=1}^{l=L}$ and $\{y_l\}_{l=1}^{l=L}$ are identically distributed squared-Nakagami variates with parameters $(k, 1/k)$ and $(m, \Omega/m)$, respectively, their first moments and variances are given by

$$\mathbb{E}[x_l] = 1 \tag{58}$$

$$\mathbb{E}[y_l] = \Omega \tag{59}$$

$$\mathbb{E}\left[x_l^2\right] - \mathbb{E}[x_l]^2 = \frac{1}{k} \tag{60}$$

$$\mathbb{E}\left[y_l^2\right] - \mathbb{E}[y_l]^2 = \frac{\Omega^2}{m} \tag{61}$$

where $l = 1, ...L$.

Due to their independency, the first moment of the the error ε can be easily calculated, according to

$$\mathbb{E}[\varepsilon] = \frac{1}{L}\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}\left(\mathbb{E}[x_i] - \mathbb{E}[x_j]\right)\left(\mathbb{E}[y_i] - \mathbb{E}[y_j]\right) = 0. \tag{62}$$

Furthermore, the variance of ε is equal to its second moment and it is derived from

$$\mathbb{E}\left[\varepsilon^2\right] = \frac{1}{L^2}\mathbb{E}\left[\left(\sum_{i=1}^{L-1}\sum_{j=i+1}^{L}(x_i - x_j)(y_i - y_j)\right)^2\right]. \tag{63}$$

After expanding the sums in (63) and taking their square, the terms that appear are calculated using (58), (59), (60) and (61). Specifically, the following terms appear,

$$\mathbb{E}\left[\left(x_i^2 - x_j^2\right)\left(x_h^2 - x_g^2\right)\left(y_i^2 - y_j^2\right)\left(y_h^2 - y_g^2\right)\right] = \begin{cases} 4\frac{\Omega^2}{km} & \text{if } i = h \text{ and } j = g \\ 0 & \text{if } i \neq h \text{ and } j \neq g \\ \frac{\Omega^2}{km} & \text{if } i = h \text{ and } j \neq g \end{cases} \tag{64}$$

where $i, j, h, g = 1...L$. After some algebra and using (64), (63) is simplified to (18).

References

[1] P. S. Bithas, N. C. Sagias, P. T. Mathiopoulos, G. K. Karagiannidis, and A. A. Rontogiannis, "On the performance analysis of digital communications over generalized-K fading channels," *IEEE Commun. Lett.*, vol. 10, no. 5, pp. 353-355, May 2006.

[2] S. Chitroub, A. Houacine, and B. Sansal, "Statistical characterization and modeling of SAR images," *Elsevier Signal Process.*, vol. 82, no. 1, Nov. 1999.

[3] E. Jakeman and P. Pusey, "A model for non-Rayleigh sea echo," *IEEE Trans. Antennas Propag.*, vol. 24, no. 6, pp. 806-814, Nov. 1976.

[4] D. A. Abraham, "Signal excess in K-distributed reverbation," *IEEE J. Ocean. Engin.*, vol. 28, no. 3, July 2003.

[5] M. A. Al-Habash, L. C. Andrews, and R. L. Philips, "Mathematical model for the irradiance probability density function of a laser beam propagating through turbulent media," *Opt. Engin.*, vol. 40, pp. 1554-1562, Aug. 2001.

[6] P. M. Shankar, "Error rates in generalized shadowed fading channels," *Wireless Personal Commun.*, vol. 28, no. 4, pp. 233-238, Feb. 2004.

[7] M. Uysal, "Diversity analysis of space-time coding in cascaded Rayleigh fading channels," *IEEE Commun. Lett.*, vol. 10, no. 3, pp. 165-167, Mar. 2006.

[8] J. S. Salo, H. M. El-Sallabi, and P. Vainikainen, "Impact of double-Rayleigh fading on system performance," in *Proc. Int. Symp. Wireless Pervasive Comput.*, 2006.

[9] L. Andrews, R. L. Philips, and C. Y. Hopen, *Laser Beam Scintillation with Applications*. SPIE Press, 2001.

[10] T. A. Tsiftsis, "Performance of heterodyne wireless optical communication systems over gamma-gamma atmospheric turbulence channels," *IEE Electron. Lett.*, vol. 44, no. 5, Feb. 2008.

[11] A. Laourine, M.-S. Alouini, and S. Affes, "On the capacity of generalized-K fading channels," *IEEE Trans. Wireless Commun.*, vol. 7, no. 7, pp. 2441-2445, July 2008.

[12] M. Uysal, S. M. Navidpour, and J. Li, "Error rate performance of coded free-space optical links over strong turbulence channels," *IEEE Commun. Lett.*, vol. 8, no. 10, pp. 635-637, Oct. 2004.

[13] P. M. Shankar, "Performance analysis of diversity combining algorithms in shadowed fading channels," *Wireless Personal Commun.*, vol. 37, no. 1-2, pp. 61-72, Apr. 2006.

[14] P. S. Bithas, P. T. Mathiopoulos, and S. A. Kotsopoulos, "Diversity reception over generalized-K (K_G) fading models," *IEEE Trans. Wireless Commun.*, vol. 6, no. 12, pp. 4238-4243, Dec. 2007.

[15] E. Bayaki, R. Schober, and R. K. Mallik, "Performance analysis of free-space optical sytems in Gamma-Gamma fading," in *IEEE Global Commun. Conf.*, 2008.

[16] J. C. S. S. Filho and M. D. Yacoub, "Simple precise approximations to Weibull sums," *IEEE Commun. Lett.*, vol. 10, no. 8, pp. 614-616, Aug. 2006.

[17] ——, "Nakagami-m approximation to the sum of M non-identical independent Nakagami-m variates," *Electron. Lett.*, vol. 40, no. 15, pp. 951-952, July 2004.

[18] ——, "Highly accurate κ-μ approximation to the sum of M independent non-identical Ricean variates," *Electron. Lett.*, vol. 41, no. 6, pp. 338-339, Mar. 2005.

[19] ——, "Highly accurate η-μ approximation to the sum of M independent non-identical Hoyt variates," *IEEE Antennas Wireless Propag. Lett.*, vol. 4, pp. 436-438, 2005.

[20] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th edition. Academic, 2000.

[21] M. Nakagami, "The m-distribution–a general formula of intensity distribution of rapid fading," *Statistical Methods Radio Wave Propag.*, vol. 40, pp. 757-768, Nov. 1962.

[22] G. K. Karagiannidis, T. H. Tsiftsis, and R. K. Mallik, "Bounds for multihop relayed communications in Nakagami-m fading," *IEEE Trans. Commun.*, vol. 54, no. 1, pp. 18-22, Jan. 2006.

[23] E. K. Al-Hussaini and A. A. M. Al-Bassiouni, "Performance of MRC diversity systems for the detection of signals with Nakagami fading," *IEEE Trans. Commun.*, vol. 33, no. 12, pp. 1315-1319, Dec. 1985.

[24] G. A. Seber and C. J. Wild, *Nonlinear Regression*. Wiley, 1989.

[25] G. K. Karagiannidis, N. C. Sagias, and T. A. Tsiftsis, "Closed-form statistics for the sum of squared Nakagami-m variates and its applications," *IEEE Trans. Commun.*, vol. 54, no. 8, pp. 1453-1458, 2006.

[26] A. Papoulis, *Probability, Random Variables, and Stochastic Procceses*, 3rd edition. McGraw-Hill, 1991.

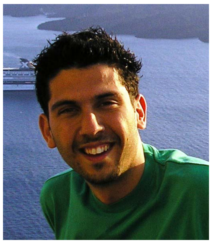
[27] M. Safari and M. Uysal, "Do we really need space-time coding for free-space optical communication with direct detection?" *IEEE Trans. Wireless Commun.*, vol. 7, no. 11, pp. 4445-4448, Nov. 2008.

[28] S. M. Navidpour and M. Uysal, "BER performance of free-space optical transmission with spatial diversity," *IEEE Trans. Wireless Commun.*, vol. 6, no. 8, pp. 2813-2819, Aug. 2007.

[29] E. Lee and V. Chan, "Part 1: optical communication over the clear turbulent atmospheric channel using diversity," *IEEE J. Sel. Areas Commun.*, vol. 22, no. 9, pp. 1896-1906, Nov. 2004.

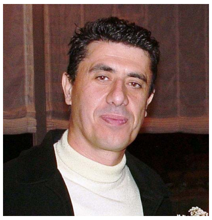
[30] T. A. Tsiftsis, H. G. Sandalidis, G. K. Karagiannidis, and M. Uysal, "Optical wireless links with spatial diversity over strong atmospheric turbulence channels," *IEEE Trans. Wireless Commun.*, vol. 8, no. 2, pp. 951-957, Feb. 2009.

[31] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integral and Series. Vol. 3: More Special Functions*. Gordon and Breach Science Publishers, 1986.



Nestor D. Chatzidiamantis (S'08) was born in Los Angeles, USA, in 1981. He received the Diploma degree (5 years) in Electrical and Computer Engineering from the Aristotle University of Thessaloniki, Greece, and the M.Sc. award (with Distinction) in Telecommunication Networks and Software from the University of Surrey, U.K. in 2005 and 2006, respectively. In November 2007, he joined again the ECE department of the Aristotle University, where he is pursuing a Ph.D degree.

His research areas span performance analysis of wireless communication systems over fading channels, communications theory and free-space optical communications.



George K. Karagiannidis (M'97-SM'04) was born in Pithagorion, Samos Island, Greece. He received the University Diploma (5 years) and Ph.D degree, both in electrical and computer engineering, from the University of Patras, in 1987 and 1999, respectively. From 2000 to 2004, he was a Senior Researcher at the Institute for Space Applications and Remote Sensing, National Observatory of Athens, Greece. In June 2004, he joined Aristotle University of Thessaloniki, Thessaloniki, where he is currently an Associate Professor of Digital Communications Systems in the Electrical and Computer Engineering Department and Head of the Telecommunications Systems and Networks Lab.

His current research interests are in the broad area of digital communications systems with emphasis on cooperative communication, adaptive modulation, MIMO systems, optical wireless and underwater communications.

He is the author or co-author of more than 110 technical papers published in scientific journals and presented at international conferences. He is also a co-author of three chapters in books and author of the Greek edition of a book on Telecommunications Systems.

Dr. Karagiannidis has been a member of Technical Program Committees for several IEEE conferences as ICC, GLOBECOM, etc. He is a member of the editorial board of the IEEE TRANSACTIONS ON COMMUNICATIONS, Senior Editor of IEEE COMMUNICATIONS LETTERS, and Lead Guest Editor of the special issue on "Optical Wireless Communications" of the IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS. He is co-recipient of the Best Paper Award of the Wireless Communications Symposium (WCS) in IEEE International Conference on Communications (ICC'07), Glasgow, U.K., June 2007.

Dr. Karagiannidis is the Chair of the IEEE COMSOC Greek Chapter.