Mean level signal crossing rate for an arbitrary stochastic process: comment

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In J. Opt. Soc. Am. A **27**, 797 (2010), Yura and Hanson derived what they claim to be "a general expression for the mean level crossing rate for an arbitrary" random process following any desired probability distribution function. On the other hand, the authors themselves assert that in some cases their results "differ somewhat from the result reported in the literature." This discrepancy "remains unexplained" by the authors "and is laid open for future discussion." In this note, we explain the reason for such discrepancy and show that the Yura-Hanson formula is indeed a special-case solution. A more general solution is then given that is applicable to arbitrary random processes and that is fully consistent with the Rice mean level crossing rate formula.

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1. INTRODUCTION

In [1], Yura and Hanson derived what they claim to be a general formula for the mean level crossing rate (LCR) of an arbitrary stationary differentiable random process $Y(t)$ with any given probability density function (PDF) $f_Y(\cdot)$. According to their derivation, the bidirectional (upward + downward crossings) LCR at level y, $\nu_Y(y)$, is given by [1, Eq. (20)]

$$\nu_Y(y) = \frac{\sigma_{\dot{y}}}{\pi\sqrt{\gamma}} \exp\left(-\frac{h^2(y)}{2}\right), \tag{1a}$$

where

- $\sigma_{\dot{y}}^2$ is the variance of the time derivative of $Y(t)$, $\dot{Y}(t)$, which, as known, can be calculated in terms of either the autocorrelation function of $Y(t)$, $R_Y(\cdot)$, or the corresponding power spectral density $S_Y(\cdot)$ as [2]

$$\sigma_{\dot{y}}^2 = -\ddot{R}_Y(\tau)|_{\tau=0} = \frac{\int_0^\infty \omega^2 S_Y(\omega)\mathrm{d}\omega}{\int_0^\infty S_Y(\omega)\mathrm{d}\omega}, \tag{1b}$$

where $\ddot{R}_Y(\tau)$ denotes the second derivative of $R_Y(\tau)$;

- $h(\cdot)$ is defined in terms of the cumulative distribution function (CDF) of $Y(t)$, $F_Y(\cdot)$, as

$$h(y) \triangleq \sqrt{2}\,\mathrm{erf}^{-1}(2F_Y(y) - 1), \tag{1c}$$

with $\mathrm{erf}^{-1}(\cdot)$ being the inverse error function;

- and γ is defined in terms of $f_Y(\cdot)$ and the first derivative of $h(\cdot)$, $\dot{h}(\cdot)$, as

$$\gamma \triangleq \int_{\mathcal{Y}} \frac{f_Y(y)}{\dot{h}^2(y)}\mathrm{d}y, \tag{1d}$$

with $\mathcal{Y}$ being the support of $f_Y(\cdot)$.

Yura and Hanson illustrated the use of (1a) by specializing it to particular processes of interest, including processes whose LCRs are already available in the literature. However, for some of these processes, the use of Yura-Hanson formula led to results that diverge somewhat from those previously available. The authors recognized such discrepancy in the paper, but they could not explain it. We explain it next and show that the Yura-Hanson formula is indeed a special-case solution for the LCR problem.

2. THE YURA-HANSON FORMULA: A SPECIAL-CASE SOLUTION

The analytical framework to compute the LCR of an arbitrary stationary differentiable random process was introduced by Rice in his pioneering work [3]. He showed that the bidirectional LCR of $Y(t)$ depends on the joint PDF of $Y(t)$ and its time derivative $\dot{Y}(t)$, $f_{Y,\dot{Y}}(\cdot,\cdot)$, as follows [3]:

$$\nu_Y(y) = \int_{-\infty}^{+\infty} |\dot{y}| f_{Y,\dot{Y}}(y,\dot{y})\mathrm{d}\dot{y}. \tag{2}$$

This way, given a process of interest, the task is to find $f_{Y,\dot{Y}}(\cdot,\cdot)$, plug it into (2), and solve the integral. For many processes, however, $f_{Y,\dot{Y}}(\cdot,\cdot)$ is not known, so that the LCR cannot be analytically determined.

In [1], Yura and Hanson proposed an artifice to find $f_{Y,\dot{Y}}(\cdot,\cdot)$ for an arbitrary random process. The approach is based on a memoryless transformation that converts the random process under investigation $Y(t)$ into a random process $X(t)$ with standard (zero-mean, unity-variance) normal PDF, namely,

$$X(t) = F_X^{-1}(F_Y(Y(t))) \triangleq h(Y(t)), \tag{3}$$

where $F_X^{-1}(\cdot)$ is the inverse standard normal CDF. Note in (3) that, for convenience, we have defined a function $h(\cdot)$ in terms of $F_Y(\cdot)$ and $F_X^{-1}(\cdot)$. Because $X(t)$ has a standard normal PDF, then $F_X^{-1}(u) = \sqrt{2}\,\mathrm{erf}^{-1}(2u-1)$, so that $h(\cdot)$ can be obtained as in (1c). The approach in (3) is indeed the well-established percentile transformation method (also known as inversion method) for generating random samples with CDF $F_X(\cdot)$ from random samples with CDF $F_Y(\cdot)$ [2, Eq. (7–157)]. The transformation not only ensures that the output process $X(t)$ has the desired CDF $F_X(\cdot)$ but also connects every higher-order statistics of $X(t)$ to the corresponding statistics of the input process $Y(t)$. In particular, the joint PDF of $X(t)$ and its time derivative $\dot{X}(t)$, $f_{X,\dot{X}}(\cdot,\cdot)$, is connected to $f_{Y,\dot{Y}}(\cdot,\cdot)$ by means of [1, Eq. (6)]

$$f_{Y,\dot{Y}}(y,\dot{y}) = [\dot{h}(y)]^2 f_{X,\dot{X}}(h(y),\dot{h}(y)\dot{y}). \tag{4}$$

Therefore, using the fact that $X(t)$ is a standard normal random process, for which $f_{X,\dot{X}}(\cdot,\cdot)$ is well known [1, Eqs. (7) and (8)], we may obtain $f_{Y,\dot{Y}}(\cdot,\cdot)$ as in (4), replace it into (2), and, after the required algebraic manipulations, finally derive the LCR of $Y(t)$ as in (1a)—this is how Yura and Hanson derived it in [1].

Such a proposed methodology is indeed simple, but it relies on a premise that is not necessarily true: $X(t)$ may *not* be a normal random process. It is true that the percentile transformation in (3) ensures that the output process $X(t)$ follows a standard normal PDF, but this is a constraint that only refers to first-order statistics, and as such does not imply that $X(t)$ is a normal random process, which refers to higher-order statistics. Recall that a random process $X(t)$ is a normal random process if the samples $X(t_1), X(t_2), \ldots, X(t_k)$, are jointly normal random variables for all k, and all choices of $t_1, t_2, \ldots, t_k$ [2]. This is a much stronger constraint than the samples $X(t_1), X(t_2), \ldots, X(t_k)$ being normal random variables separately. Therefore, whether or not the output $X(t)$ is a normal random process depends on the higher-order statistical properties of the input $Y(t)$ used to create $X(t)$. We cannot conjecture on $f_{X,\dot{X}}(\cdot,\cdot)$ being that of a normal random process and then replace this into (4) in order to obtain $f_{Y,\dot{Y}}(\cdot,\cdot)$. The correct reasoning is right the opposite: depending on $f_{Y,\dot{Y}}(\cdot,\cdot)$ (the cause), $f_{X,\dot{X}}(\cdot,\cdot)$ (the effect) may or may not be that of a normal random process.

From the above, it is clear that the Yura-Hanson formula does not represent the LCR of an arbitrary random process. However, it is a valid particular-case solution, as follows. In the derivation process, Yura and Hanson assumed that the transformed process $X(t)$ is a standard normal random process. Now, consider the set of all stationary differentiable random processes with a given PDF $f_Y(\cdot)$. (The set of processes in such a condition is infinite.) Then, among all of the processes in this set, the Yura-Hanson formula provides the LCR for the particular subset of processes whose transformation via (3) produces a standard normal random process. For all of the remaining processes in the set (again, an infinite subset), the Yura-Hanson formula fails to give the correct LCR.

An alternative look at the Yura-Hanson formula is obtained by the inversion of (3):

$$Y(t) = F_Y^{-1}(F_X(X(t))), \tag{5}$$

where $F_Y^{-1}(\cdot)$ is the inverse CDF of $Y(t)$. This is a reverse look: $Y(t)$ is now the output process being produced from an input process $X(t)$ via percentile transformation. On the one hand, by design, $Y(t)$ always matches the target CDF $F_Y(\cdot)$, irrespective of the input process $X(t)$ we choose. On the other, that choice strongly affects the higher-order statistics of $Y(t)$. For example, as shown in (4), $f_{Y,\dot{Y}}(\cdot,\cdot)$ may vary, depending on the $f_{X,\dot{X}}(\cdot,\cdot)$, $h(\cdot)$, and $\dot{h}(\cdot)$ associated to the process $X(t)$ we choose. As before, consider the set of all stationary differentiable random processes $Y(t)$ with a given PDF $f_Y(\cdot)$ and generated via (3) from any arbitrary input process $X(t)$. Then, among all of the processes in this set, the Yura-Hanson formula provides the LCR for the particular subset of processes that have been generated from standard normal random processes. Again, for all of the remaining processes in the set (an infinite subset), the Yura-Hanson formula fails to give the correct LCR.

Note that the Yura-Hanson formula overlooks some statistical content that may impact the LCR. According to the Rice formula, the LCR depends on the joint PDF of the random process and its time derivative, $f_{Y,\dot{Y}}(\cdot,\cdot)$. In contrast, according to the Yura-Hanson formula, the LCR depends on the marginal PDF of the random process, $f_Y(\cdot)$, and on the variance of its time derivative, $\sigma_{\dot{y}}^2$, which in general contains less information than $f_{Y,\dot{Y}}(\cdot,\cdot)$. This reinforces the fact that the Yura-Hanson formula does not represent a general solution for the LCR problem.

As for the discrepancies reported in [1] between the Yura-Hanson formula and some previous results in the literature, there is nothing wrong with them. Such discrepancies are indeed expected. The point here is that processes with identical PDFs $f_Y(\cdot)$ may have different joint PDFs $f_{Y,\dot{Y}}(\cdot,\cdot)$ and, consequently, different LCRs—this is the Rice formula. Take, for instance, the Weibull process. It has been implemented in [1] via two different methods. On the one hand, when implemented via percentile transformation of a normal random process, which is exactly the model implicit in Yura-Hanson's analysis, then the Weibull process matches the Yura-Hanson formula, as expected. On the other, when implemented via sum of squares of two independent normal random processes [1, Eq. (36)], which is exactly the model used in Khimenko's analysis (see [3] in [1]), then the Weibull process matches the Khimenko formula. The corresponding results are shown in [1, Fig. 4]. Again, it is all about two random processes with identical PDFs but different LCRs.

3. THE GENERAL-CASE SOLUTION

Consider the generation of an arbitrary random process $Y(t)$ from another arbitrary random process $X(t)$ via the percentile transformation in (5). Assume that we know the LCR of $X(t)$, $\nu_X(\cdot)$, and that we want to find the LCR of $Y(t)$, $\nu_Y(\cdot)$. There is a simple general solution, as we show next. We have already presented this solution in [4] and reproduce it here for completeness.

First, note that the percentile transformation in (5) is bijective, which means that each level x of $X(t)$ is mapped one-to-one onto a different level y of $Y(t)$. Therefore, (i) the input process $X(t)$ crosses a given level x if and only if the output process $Y(t)$ crosses the level $F_Y^{-1}(F_X(x))$; or, equivalently, based on the inverse transformation in (3), (ii) the output process crosses a given level y if and only if the input process crosses the level $F_X^{-1}(F_Y(y)) \triangleq h(y)$. Finally, since the LCR represents the average number of crossings per unit time, it follows directly from (ii) that

$$\nu_Y(y) = \nu_X(h(y)), \tag{6}$$

where

$$h(y) \triangleq F_X^{-1}(F_Y(y)). \tag{7}$$

Our formula (6), already presented in [4], is a general solution that gives the LCR of $Y(t)$ in terms of the LCR of $X(t)$, applicable to arbitrary random processes and fully consistent with the Rice formula in all of the cases. It is mainly attractive when the LCR of $X(t)$ is available or else easier to compute than the Rice integral and the underlying $f_{Y,\dot{Y}}(\cdot,\cdot)$. In particular, when $X(t)$ is a standard normal random process, our formula specializes to the Yura-Hanson formula (1a). In this special case, $\nu_X(\cdot)$ is known to be given by [1, Eq. (19)], and, since $F_X^{-1}(u) = \sqrt{2}\,\mathrm{erf}^{-1}(2u-1)$, $h(y)$ reduces to (1c). Replacing this into (6), we obtain the Yura-Hanson formula.

A final word concerns the right choice of the input random process $X(t)$ to be used into the percentile transformation in (5). On the one hand, by design, any choice equally matches the target PDF $f_Y(\cdot)$. On the other, as we showed here, different choices of $X(t)$ differently affect the higher-order statistics of $Y(t)$. For example, the way in which $\nu_X(\cdot)$ affects $\nu_Y(\cdot)$ is given in (6). As engineers, depending on the application at hand, we should choose $X(t)$ in such a form that the resulting higher-order statistics of $Y(t)$ better describe the real physical phenomena involved with the process under investigation. We may want to choose the simplest $X(t)$ that fulfills that condition, but should not choose it simpler.

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