Novel Approximations to the Statistics of Products of Independent Random Variables and Their Applications in Wireless Communications

Yunfei Chen, *Senior Member, IEEE*, George K. Karagiannidis, *Senior Member, IEEE*, Hao Lu, and Ning Cao

***Abstract*—A novel analytical framework for evaluating the statistics of the product of independent random variables is proposed. Compared with other methods, which use either infinite series or special functions, the new method provides simple and efficient closed-form approximations, in terms of elementary functions, such as powers and exponentials. Numerical results, which are used to check the accuracy of the new approximations, show that it is quite accurate in most regions of interest. As an application, the proposed analytical results can be efficiently used in wireless communications theory to evaluate in closed form the outage probability of cascaded fading channels, as well as the rate offset of the hybrid automatic repeat request (H-ARQ) transmission. Numerical examples show that the derived expressions provide significant insights on the behaviors of important system parameters as the outage probability and the rate offset.**

***Index Terms*—Cascaded fading channels, keyhole channels, product of random variables (RVs), wireless communications, wireless relaying.**

I. Introduction

IN SEVERAL cases of wireless systems, the communication channel can be well described as a product of independent random variables (RVs). For example, in a multihop wireless relaying system using the amplify-and-forward protocol with fixed amplification factor and statistically independent hops, the cascaded channel from source to destination can be modeled as the product of the RVs that describe the channel gains for the individual hops [1]. Moreover, in a multiple-input–multiple-output (MIMO) keyhole system, the electromagnetic wave propagates through several keyholes such that the overall channel gain can also be modeled as the product of the RVs that describe the individual keyhole channels [2]. In addition to channel modeling, the product of RVs also arises in several cases, where the performances of wireless communications systems are considered. For example, the rate offset or the ϵ-outage capacity of hybrid automatic repeat request (H-ARQ) transmission at high signal-to-noise ratios (SNRs) can be well approximated as the logarithm of the product of the channel gains for the data blocks in the transmission [3], [4]. In addition, the outage probability of the received signal in a multihop wireless relaying system is a function of the product of the channel gains for all the hops [5]. To design wireless systems based on these channel models and analyze wireless systems for accurate performance prediction, the statistics of the product are required.

Previous works on the statistics of the product of independent RVs include the following: In [6], the probability density function (pdf) of the product of independent Gaussian RVs was derived in the form of an infinite series, whereas, in [7], the pdf of the same product was derived in terms of the special Meijer-G function. In addition to Gaussian, [6] and [7] also obtained pdfs for the products of Cauchy, Beta, and Gamma RVs. In [8], the pdf of the product of independent RVs was derived for the H-function distribution, which includes most of the commonly used distributions as special cases, and the results were given in the form of the special H-function. In [9], the pdf of the product of independent Rayleigh RVs was derived in terms of both the Meijer-G function and the infinite series. In [10], the pdf of the product of independent Nakagami RVs was derived using an infinite series approach, whereas, in [5], it was given in terms of the Meijer-G function. Finally, in [11], the pdf of the product of independent generalized Nakagami-m RVs was also derived in terms of the Meijer-G function.

All the aforementioned works can be categorized into two methods, i.e., the infinite series and the special function methods. However, the infinite series has to be truncated, and the number of truncated terms has to be heuristically determined for each number of independent RVs in the product, making it difficult to implement in software. On the other hand, the special function method provides closed-form results for the pdf without truncation. However, special functions such as the Meijer-G and H-functions are essentially contour integrals; thus, to calculate the pdf, one has to either solve the contour integral using mathematical software or construct a lookup table with all possible values of these functions. However, the solution of contour integral is time consuming, whereas the construction of a lookup table is memory consuming. Therefore, both the infinite series and special function methods are computationally complicated.

Manuscript received October 7, 2011; revised November 23, 2011; accepted November 24, 2011. Date of publication December 7, 2011; date of current version February 21, 2012. The review of this paper was coordinated by Dr. K. T. Wong.

Y. Chen is with the School of Engineering, University of Warwick, CV4 7AL Coventry, U.K., and also with the School of Computer and Information, Hohai University, Nanjing 210098, China (e-mail: Yunfei.Chen@warwick.ac.uk).

G. K. Karagiannidis is with the Department of Electrical and Computer Engineering, Aristotle University of Thessaloniki, 54124 Thessaloniki, Greece (e-mail: geokarag@auth.gr).

H. Lu and N. Cao are with Hohai University, Nanjing 210098, China (e-mail: toluhao@hotmail.com; caoning@vip.163.com).

Digital Object Identifier 10.1109/TVT.2011.2178441

In this paper, we propose an efficient analytical framework as a simpler method to approximately evaluate the pdf and the cumulative distribution function (cdf) of the product of independent RVs. Unlike the infinite series method, the new approximate pdf and cdf are derived in closed form, so that no truncation is required. Moreover, unlike the special function method, the new approximate pdf is expressed in terms of elementary functions, such as powers and exponentials, so that no contour integration or table lookup is needed. Several commonly used distributions in wireless communications are studied: the generalized Gamma distribution, the generalized Nakagami-m distribution, and the generalized Gaussian distribution. The generalized Gamma distribution is found to fit certain physical fading environments well [12], whereas it includes several important distributions, such as the Weibull, as special cases [13]. The generalized Nakagami-m is a generalization of the widely used Nakagami-m distribution in wireless communications. The generalized Gaussian distribution is a generalization of the widely used Gaussian model and is found to fit well certain interference models in wireless systems [14].

The accuracy of the new approximation is examined after comparison with the exact pdf and cdf evaluated via the complicated special function method. Numerical results show that the newly derived approximation works very well in most regions of interest. As an application, these new approximate pdfs and cdfs are used to calculate the outage probability of the multihop wireless relaying system and the rate offset of the H-ARQ transmission. Using these new closed-form expressions, insights on the behaviors of the outage probability and the rate offset are obtained that are otherwise not possible using the existing infinite series or special function methods.

The rest of this paper is organized as follows: In Section II, the distributions of generalized Gamma, generalized Nakagami-m, and generalized Gaussian are introduced. Section III obtains the new approximations to the statistics of the products of generalized Gamma, generalized Nakagami-m, and generalized Gaussian RVs. In Section IV, the outage probability and the rate offset are derived based on the new approximations. In Section V, numerical results are presented to show the accuracy of the new approximations and the wireless applications. Finally, concluding remarks are made in Section VI.

II. System Model

Consider the product

$$Y'_s = X'_{1,s}X'_{2,s}\cdots X'_{n,s} \tag{1}$$

where $X'_{1,s}, X'_{2,s}, \ldots, X'_{n,s}$, are n independent RVs, and s indicates the specific name of the distribution.

For the generalized Gamma distribution, one has the pdf of $X'_{i,GG}$ as [12]

$$f_{X'_{i,GG}}(x) = \frac{vm^m}{\Gamma(m)\Omega_i^{mv}} x^{mv-1} e^{-\frac{mx^v}{\Omega_i^v}}, \qquad x > 0 \tag{2}$$

where $i = 1, 2, \ldots, n$, $\Gamma(\cdot)$ is the complete Gamma function, GG denotes the generalized Gamma distribution, $m > 0.5$ is the fading figure, $\Omega_i > 0$ is the scale parameter, and $v > 0$ is the shape parameter. By normalizing the RV with respect to $(\Omega_i/m^{(1/v)})$ as $X_{i,GG} = (X'_{i,GG}/(\Omega_i/m^{(1/v)}))$, one has

$$f_{X_{i,GG}}(x) = \frac{v}{\Gamma(m)} x^{mv-1} e^{-x^v}, \qquad x > 0. \tag{3}$$

When $m = 1$, (3) corresponds to the Weibull distribution.

For the generalized Nakagami-m distribution, one has the pdf of $X'_{i,GN}$ as [11]

$$f_{X'_{i,GN}}(x) = \frac{2v}{\Gamma(m)} \left(\frac{\beta}{\Omega_i}\right)^{mv} x^{2mv-1} e^{-\left(\frac{\beta}{\Omega_i}\right)^v x^{2v}}, \qquad x > 0 \tag{4}$$

where GN denotes the generalized Nakagami-m distribution, $\Omega_i > 0$ is the average power, $m > 0.5$ is the fading figure, $v > 0$ is the shape parameter, and $\beta = (\Gamma(m + (1/v))/\Gamma(m))$.

By normalizing the RV with respect to $\sqrt{(\Omega_i/\beta)}$ as $X_{i,GN} = (X'_{i,GN}/\sqrt{(\Omega_i/\beta)})$, one has

$$f_{X_{i,GN}}(x) = \frac{2v}{\Gamma(m)} x^{2mv-1} e^{-x^{2v}}, \qquad x > 0. \tag{5}$$

For the generalized Gaussian distribution, one has the pdf of $X'_{i,GA}$ as [15]

$$f_{X'_{i,GA}}(x) = \frac{p\sqrt{\Gamma\left(\frac{3}{p}\right)}}{2\sigma_i\Gamma\left(\frac{1}{p}\right)\sqrt{\Gamma\left(\frac{1}{p}\right)}} e^{-\frac{|x|^p}{\sigma_i^p}\left(\frac{\Gamma\left(\frac{3}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}\right)^{\frac{p}{2}}} \tag{6}$$

for $+\infty > x > -\infty$, where GA denotes the generalized Gaussian distribution, p is the shape parameter, σ_i^2 is the average power, and the location parameter has been omitted for simplicity.

If one normalizes X'_i with respect to $(\sigma_i/2^{(1/p)}\sqrt{(\Gamma(3/p)/\Gamma(1/p))})$ as $X_{i,GA} = X'_{i,GA} * (2^{(1/p)}\sqrt{(\Gamma(3/p)/\Gamma(1/p))}/\sigma_i)$, one has

$$f_{X_{i,GA}}(x) = \frac{p}{2^{1+\frac{1}{p}}\Gamma\left(\frac{1}{p}\right)} e^{-\frac{1}{2}|x|^p}, \qquad +\infty > x > -\infty. \tag{7}$$

For later use, the product of the independent normalized RVs is defined as

$$Y_s = X_{1,s}X_{2,s}\cdots X_{n,s}. \tag{8}$$

In the previous discussion, it is assumed that different RVs have different scale parameters or average powers while having the same shape parameters. Note that, from (3) and (5), the generalized Nakagami-m RV is the square root of the generalized Gamma RV. In some wireless applications, the generalized Nakagami-m RV describes the distribution of the signal amplitude, whereas the generalized Gamma RV describes the distribution of the signal power. Both amplitude and power are of great interest in wireless communications. Therefore, they are separately discussed in this paper. Note also that the generalized Gaussian RV can take any real value, whereas the generalized Nakagami-m and Gamma RVs only take positive values. Consequently, the product of generalized Gaussian RVs can take any real value, whereas the products of generalized Nakagami-m and Gamma RVs only take positive values.

III. New Efficient Approximations

Starting from the exponential RV, the pdf of the product of independent normalized exponential RV can be derived using a method similar to that in [6] and [7]. The Mellin transform of function $f(x)$ is defined as

$$M\{s\} = E\{X^{s-1}\} = \int_0^\infty x^{s-1} f(x)dx \tag{9}$$

where $E\{\cdot\}$ represents the expectation operation, and $X > 0$. Consider the normalized exponential RV with pdf

$$f_{X_{i,EX}}(x) = e^{-x}, \qquad x > 0. \tag{10}$$

Its Mellin transform is given by

$$M_{X_{i,EX}}(s) = \Gamma(s). \tag{11}$$

A useful characteristic is that the Mellin transform of the product of independent RVs is equal to the product of the Mellin transforms of the individual RVs. Therefore, the Mellin transform of the product of n independent normalized exponential RVs can be derived from (11) as

$$M_{Y_{EX}}(s) = \Gamma^n(s). \tag{12}$$

From (12), the exact pdf of the product of n independent normalized exponential RVs is derived by taking an inverse Mellin transform as

$$f_{Y_{EX}}(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma^n(s)ds \tag{13}$$

where the path of integration is any line parallel to the imaginary axis lying within the strip of analyticity of $\Gamma^n(s)$. The contour integral in (13) cannot be solved in closed form without using an infinite series or a special function. In the following, we will derive an approximation to (13). However, it is very difficult to directly derive an approximation from the transform method in (13). Therefore, to overcome this difficulty, we will use the pdf method instead and then equate the results from the two methods.

It was shown in [16] that, for the nth-order root of the product of independent Rayleigh RVs, if

$$Z'_{RA} = \left(X'_{1,RA}X'_{2,RA}\cdots X'_{n,RA}\right)^{\frac{1}{n}} \tag{14}$$

where RA denotes the Rayleigh distribution and $X'_{i,RA}$ is the ith Rayleigh RV with distribution

$$f_{X'_{i,RA}}(x) = xe^{-\frac{x^2}{2}}, \qquad x > 0 \tag{15}$$

then the pdf of Z'_{RA} can be well approximated as a Nakagami-m distribution with

$$f_{Z'_{RA}}(x) \approx 2\left(\frac{m_0}{\Omega_0}\right)^{m_0} \frac{1}{\Gamma(m_0)} x^{2m_0-1} e^{-\frac{m_0}{\Omega_0}x^2}, \qquad x > 0 \tag{16}$$

where $m_0 = 0.6102 * n + 0.4263$ and $\Omega_0 = 0.8808 * n^{-0.9661} + 1.12$ are heuristically determined using the MATLAB distribution fitting tool, with 10^6 simulated values for Z'_{RA}. Since Rayleigh RV is related to exponential RV, we can use (16) to derive an approximation to the pdf of the product of exponential RVs. Let $X_{i,RA} = (X'_{i,RA}/\sqrt{2})$. Therefore, the approximate pdf of

$$Z_{RA} = (X_{1,RA}X_{2,RA}\cdots X_{n,RA})^{\frac{1}{n}} \tag{17}$$

is

$$f_{Z_{RA}}(x) \approx 2\left(\frac{2m_0}{\Omega_0}\right)^{m_0} \frac{1}{\Gamma(m_0)} x^{2m_0-1} e^{-\frac{2m_0}{\Omega_0}x^2}, \qquad x > 0 \tag{18}$$

where $X_{i,RA}$ follows the normalized Rayleigh distribution of

$$f_{X_{i,RA}}(x) = 2xe^{-x^2}, \qquad x > 0 \tag{19}$$

and m_0 and Ω_0 are defined as before. Let

$$Y_{EX} = X_{1,EX}X_{2,EX}\cdots X_{n,EX} \tag{20}$$

where EX denotes the exponential distribution, and $X_{i,EX}$ follows a normalized exponential distribution in (10). It is easy to find from (10) and (19) that $X_{i,EX} = X^2_{i,RA}$. Since both RVs take positive values, one has, from (17) and (20)

$$Z_{RA} = (X_{1,EX}X_{2,EX}\cdots X_{n,EX})^{\frac{1}{2n}} = Y_{EX}^{\frac{1}{2n}}. \tag{21}$$

Using (18), (21), and the pdf of a function of RV $f_Y(y) = f_X(g(y))(dx/dy)$ with $X = g(Y) = Y^{(1/2n)}$ in this case, the approximate pdf of the product of independent normalized exponential RVs is

$$f_{Y_{EX}}(x) \approx \left(\frac{2m_0}{\Omega_0}\right)^{m_0} \frac{1}{n\Gamma(m_0)} x^{\frac{m_0}{n}-1} e^{-\frac{2m_0}{\Omega_0}x^{\frac{1}{n}}}, \qquad x > 0. \tag{22}$$

Since both (13) and (22) calculate the same function, by equating them, an approximation to the contour integral in (13) is

$$\begin{aligned} f_{Y_{EX}}(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma^n(s)ds \\ &\approx \left(\frac{2m_0}{\Omega_0}\right)^{m_0} \frac{1}{n\Gamma(m_0)} x^{\frac{m_0}{n}-1} e^{-\frac{2m_0}{\Omega_0}x^{\frac{1}{n}}}, \qquad x > 0. \end{aligned} \tag{23}$$

In the following, (23) will be used to approximate the pdfs of the generalized Gamma, the generalized Nakagami, and the generalized Gaussian distributions.

A. Generalized Gamma Distribution

Using the Mellin transform and the approximation in (23), the approximation to the pdf of the product of n independent normalized generalized Gamma RVs can be derived as (for the proof see the Appendix)

$$f_{Y_{GG}}(x) \approx \frac{v}{n\Gamma(m_0+mn-n)}\left(\frac{2m_0}{\Omega_0}\right)^{m_0+mn-n} \times x^{\frac{vm_0}{n}+mv-v-1}e^{-\frac{2m_0}{\Omega_0}x^{\frac{v}{n}}}, \qquad x>0. \quad (24)$$

The approximation to the cdf of the product of n independent normalized generalized Gamma RVs can be derived by integrating (24) over x from 0 to y as

$$F_{Y_{GG}}(y) \approx \gamma\left(m_0+mn-n, \frac{2m_0}{\Omega_0}y^{\frac{v}{n}}\right) \quad (25)$$

where

$$\gamma(a,x) = \frac{1}{\Gamma(a)}\int_0^x e^{-t}t^{a-1}dt \quad (26)$$

is the lower incomplete Gamma function. Note that (26) is different from what is defined in [21, eq. (8.350.1)] because of the extra normalization over $\Gamma(a)$.

Since one has $Y'_{GG} = Y_{GG}((\prod_{i=1}^n \Omega_i)/m^{(n/v)})$, the new approximation to the pdf of the product of n independent unnormalized generalized Gamma RVs is

$$f_{Y'_{GG}}(x) \approx \frac{vx^{\frac{vm_0}{n}+mv-v-1}}{n\Gamma(m_0+mn-n)}\left(\frac{2m_0}{\Omega_0}\right)^{m_0+mn-n} \times \left(\frac{m^{\frac{n}{v}}}{\prod_{i=1}^n \Omega_i}\right)^{\frac{vm_0}{n}+mv-v} e^{-m\frac{2m_0}{\Omega_0}\left(\frac{x}{\prod_{i=1}^n \Omega_i}\right)^{\frac{v}{n}}} \quad (27)$$

where $x>0$, and the new approximation to the cdf of the product of n independent unnormalized generalized Gamma RVs can be derived as

$$F_{Y'_{GG}}(y) \approx \gamma\left(m_0+mn-n, m\frac{2m_0}{\Omega_0}\left(\frac{y}{\prod_{i=1}^n \Omega_i}\right)^{\frac{v}{n}}\right). \quad (28)$$

It is evident from (27) that the new approximation leads to much simpler evaluation of the pdf of the product of generalized Gamma RVs than the infinite series and special function methods. Furthermore, the new approximation in (28) gives the cdf of the product of independent generalized Gamma RVs in terms of the incomplete Gamma function. From (26), this function can be calculated as an integral with finite limits over the product of power and exponential functions. This function is also a built-in function in most commonly used software.

A special case occurs when $m=1$. In this case, one has new approximations to the pdf and the cdf of the product of independent normalized Weibull RVs as

$$f_{Y_{WB}}(x) \approx \frac{v}{n\Gamma(m_0)}\left(\frac{2m_0}{\Omega_0}\right)^{m_0} x^{\frac{vm_0}{n}-1}e^{-\frac{2m_0}{\Omega_0}x^{\frac{v}{n}}}, \qquad x>0 \quad (29)$$

$$F_{Y_{WB}}(y) \approx \gamma\left(m_0, \frac{2m_0}{\Omega_0}y^{\frac{v}{n}}\right) \quad (30)$$

respectively, where WB denotes the Weibull distribution. The new approximations to the pdf and cdf of the product of arbitrary Weibull RVs are

$$f_{Y'_{WB}}(x) \approx \frac{vx^{\frac{vm_0}{n}-1}}{n\Gamma(m_0)}\left(\frac{2m_0}{\Omega_0}\right)^{m_0}\left(\frac{1}{\prod_{i=1}^n \Omega_i}\right)^{\frac{vm_0}{n}} \times e^{-\frac{2m_0}{\Omega_0}\left(\frac{x}{\prod_{i=1}^n \Omega_i}\right)^{\frac{v}{n}}}, \qquad x>0 \quad (31)$$

$$F_{Y'_{WB}}(y) \approx \gamma\left(m_0, \frac{2m_0}{\Omega_0}\left(\frac{y}{\prod_{i=1}^n \Omega_i}\right)^{\frac{v}{n}}\right) \quad (32)$$

respectively. The Weibull distribution is an important multipath fading model, particularly for radio systems operating in the 800/900-MHz range [18].

B. Generalized Nakagami-m Distribution

Using a method similar to before, one can derive the pdf of the product of normalized generalized Nakagami-m RVs in the Appendix as

$$f_{Y_{GN}}(x) \approx \frac{2v}{n\Gamma(m_0+mn-n)}\left(\frac{2m_0}{\Omega_0}\right)^{m_0+mn-n} \times x^{\frac{2vm_0}{n}+2mv-2v-1}e^{-\frac{2m_0}{\Omega_0}x^{\frac{2v}{n}}}, \qquad x>0. \quad (33)$$

By integrating (33) over x from 0 to y, the approximate cdf can be derived as

$$F_{Y_{GN}}(y) \approx \gamma\left(m_0+mn-n, \frac{2m_0}{\Omega_0}y^{\frac{2v}{n}}\right). \quad (34)$$

Since $Y'_{GN} = Y_{GN}\prod_{i=1}^n(\sqrt{(\Omega_i/\beta)})$, one has

$$f_{Y'_{GN}}(x) \approx \frac{2vx^{\frac{2vm_0}{n}+2mv-2v-1}}{n\Gamma(m_0+mn-n)}\left(\frac{2m_0}{\Omega_0}\right)^{m_0+mn-n} \times \left(\frac{\beta^n}{\prod_{i=1}^n \Omega_i}\right)^{\frac{vm_0}{n}+mv-v} e^{-\frac{2m_0}{\Omega_0}\left(\frac{\beta^{\frac{n}{2}}x}{\prod_{i=1}^n \Omega_i^{\frac{1}{2}}}\right)^{\frac{2v}{n}}} \quad (35)$$

with $x>0$ and

$$F_{Y'_{GN}}(y) \approx \gamma\left(m_0+mn-n, \frac{2m_0}{\Omega_0}\left(\frac{\beta^{\frac{n}{2}}y}{\prod_{i=1}^n \Omega_i^{\frac{1}{2}}}\right)^{\frac{2v}{n}}\right). \quad (36)$$

For the special case of Rayleigh RV, one has $m=v=1$ in (35) and (36) to give the approximate pdf and cdf of the product of Rayleigh RVs as

$$f_{Y'_{GN}}(x) \approx \frac{2x^{\frac{2m_0}{n}-1}}{n\Gamma(m_0)}\left(\frac{2m_0}{\Omega_0}\right)^{m_0}\left(\frac{\beta^n}{\prod_{i=1}^n \Omega_i}\right)^{\frac{m_0}{n}} \times e^{-\frac{2m_0}{\Omega_0}\left(\frac{\beta^{\frac{n}{2}}x}{\prod_{i=1}^n \Omega_i^{\frac{1}{2}}}\right)^{\frac{2}{n}}}, \quad x > 0 \tag{37}$$

$$F_{Y'_{GN}}(y) \approx \gamma\left(m_0, \frac{2m_0}{\Omega_0}\left(\frac{\beta^{\frac{n}{2}}y}{\prod_{i=1}^n \Omega_i^{\frac{1}{2}}}\right)^{\frac{2}{n}}\right) \tag{38}$$

respectively. Again, these approximate pdfs are expressed in terms of simple functions and therefore are very easy to calculate.

C. Generalized Gaussian Distribution

The approximate pdf of the product of independent normalized generalized Gaussian RVs is derived in the Appendix as

$$f_{Y_{GA}}(x) \approx \frac{p}{2n\Gamma\left(m_0 - n + \frac{n}{p}\right)}\left(\frac{m_0}{\Omega_0}\right)^{m_0-n+\frac{n}{p}} \times |x|^{\frac{pm_0}{n}-p} e^{-\frac{m_0}{\Omega_0}|x|^{\frac{p}{n}}}, \quad +\infty > x > -\infty. \tag{39}$$

Note that, when $m_0 < n$, the value of the pdf in (39) goes to infinity when x approaches zero, which agrees with that observed in [7] for the product of two Gaussian RVs, whose pdf is a modified Bessel function of the second kind with zeroth order. By integrating $f_{Y_{GA}}(x)$ over x from $-\infty$ to y, one has the cdf of the product of independent normalized generalized Gaussian RVs as

$$F_{Y_{GA}}(y) \approx 0.5 + 0.5 \text{ sign } (y)\gamma\left(m_0 - n + \frac{n}{p}, \frac{m_0}{\Omega_0}|y|^{\frac{p}{n}}\right) \tag{40}$$

where $\text{sign}(\cdot)$ is the signum function. Since $Y'_{GA} = Y_{GA} * (\prod_{i=1}^n \sigma_i / 2^{(n/p)}) * ((\Gamma(1/p)/\Gamma(3/p)))^{(n/2)}$, the new approximation to the pdf of the product of n independent generalized Gaussian RVs can be derived as

$$f_{Y'_{GA}}(x) \approx \frac{p|x|^{\frac{pm_0}{n}-p}}{n\Gamma(m_0 - n + \frac{n}{p})}\left(\frac{2m_0}{\Omega_0}\right)^{m_0-n+\frac{n}{p}} \times \frac{\left(\frac{\Gamma\left(\frac{3}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}\right)^{\frac{pm_0}{2}-\frac{pn}{2}+\frac{n}{2}}}{\left(\prod_{i=1}^n \sigma_i\right)^{\frac{pm_0}{n}-p+1}} e^{-\frac{m_0}{\Omega_0}\frac{2\left(\frac{\Gamma\left(\frac{3}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}\right)^{\frac{p}{2}}}{\left(\prod_{i=1}^n \sigma_i\right)^{\frac{p}{n}}}|x|^{\frac{p}{n}}} \tag{41}$$

for $+\infty > x > -\infty$, and the new approximation to the cdf of the product of n independent generalized Gaussian RVs is

$$F_{Y'_{GA}}(y) \approx 0.5 + 0.5\text{sign}(y) \times \gamma\left(m_0 - n + \frac{n}{p}, \frac{m_0}{\Omega_0}\frac{2\left(\frac{\Gamma\left(\frac{3}{p}\right)}{\Gamma\left(\frac{1}{p}\right)}\right)^{\frac{p}{2}}}{\left(\prod_{i=1}^n \sigma_i\right)^{\frac{p}{n}}}y^{\frac{p}{n}}\right). \tag{42}$$

In the next section, several applications of the approximate pdfs and cdfs will be discussed.

IV. Applications

In this section, the approximate pdfs and cdfs derived in the previous section are adopted to calculate the outage probability for wireless relaying with cascaded fading channels and the rate offset for H-ARQ transmission.

A. Outage Probability

For multihop wireless relaying systems, the end-to-end SNR is given by [5, eq. (11)]

$$\alpha = \frac{E_s}{N_0}\prod_{i=1}^n \left|X'_{i,s}\right|^2 \tag{43}$$

where (E_s/N_0) is the transmitted SNR, and $X'_{i,s}$ represents the fading coefficient of the ith hop. In (43), E_s is the transmitted signal power at the source, whose signal is relayed and suffers from fadings in n relays, with the fading coefficient of the ith relay being $X'_{i,s}$ and the fading power of the ith relay being $|X'_{i,s}|^2$, so that the received signal power at the destination is $E_s \prod_{i=1}^n |X'_{i,s}|^2$ and the received SNR at the destination is α. Define

$$\bar{\alpha} = \frac{E_s}{N_0}\prod_{i=1}^n E\left\{\left|X'_{i,s}\right|^2\right\} \tag{44}$$

as the average end-to-end SNR. By normalizing (43) with respect to the average end-to-end SNR in (44), one has

$$\tilde{\alpha} = \prod_{i=1}^n \frac{\left|X'_{i,s}\right|^2}{E\left\{\left|X'_{i,s}\right|^2\right\}}. \tag{45}$$

The outage probability is defined as the probability that the SNR of α is below a certain threshold as $P_o = Pr\{\alpha < \alpha_0\}$ [5]. By normalizing both sides of the inequality with the positive value of $\bar{\alpha}$ and letting $\alpha_{\text{th}} = (\alpha_0/\bar{\alpha})$, one also has

$$P_o = Pr\{\tilde{\alpha} < \alpha_{\text{th}}\}. \tag{46}$$

Similar analyses for other channel models have been conducted in [22] and [23]. We derive it for generalized Gamma, generalized Nakagami-m, and generalized Gaussian channels in this paper. For the generalized Gamma fading channel, one has

$$E\left\{\left|X'_{i,GG}\right|^2\right\} = \frac{\Gamma\left(m + \frac{2}{v}\right)}{\Gamma(m)}\left(\frac{\Omega_i}{m^{\frac{1}{v}}}\right)^2 \tag{47}$$

$$\tilde{\alpha} = \frac{\prod_{i=1}^n \left|X'_{i,s}\right|^2}{\frac{\Gamma^n\left(m+\frac{2}{v}\right)}{\Gamma^n(m)}\left(\frac{\prod_{i=1}^n \Omega_i}{m^{\frac{n}{v}}}\right)^2}. \tag{48}$$

Thus

$$P_o = Pr\left\{Y'_{GG} < \sqrt{\alpha_{\text{th}}}\frac{\Gamma^{\frac{n}{2}}\left(m + \frac{2}{v}\right)}{\Gamma^{\frac{n}{2}}(m)}\frac{\prod_{i=1}^n \Omega_i}{m^{\frac{n}{v}}}\right\}. \tag{49}$$

Using the pdf of Y'_{GG} in (27) or the cdf of Y'_{GG} in (28) and after some mathematical manipulations, one has

$$P_o = \gamma \left(m_0 + mn - n, \frac{2m_0}{\Omega_0} \alpha_{\text{th}}^{\frac{v}{2n}} \frac{\Gamma^{\frac{v}{2}}\left(m + \frac{2}{v}\right)}{\Gamma^{\frac{v}{2}}(m)} \right). \tag{50}$$

One sees that the new approximate pdf and the new approximate cdf allow us to have a closed-form expression for the outage probability. The incomplete Gamma function only involves with a 1-D real integral with finite limits, unlike the Meijer-G function and the H-function that involve with a complex contour integral with infinite limits. Therefore, it is relatively simple to calculate or tabulate. Similarly, the outage probabilities for the generalized Nakagami-m fading channel and the generalized Gaussian fading channel can be derived as

$$P_o = \gamma \left(m_0 + mn - n, \frac{2m_0}{\Omega_0} \alpha_{\text{th}}^{\frac{v}{n}} \frac{\Gamma^{v}\left(m + \frac{1}{v}\right)}{\Gamma^{v}(m)} \right) \tag{51}$$

$$P_o = \gamma \left(\frac{n}{p}, \frac{2m_0}{\Omega_0} \alpha_{\text{th}}^{\frac{p}{2n}} \frac{\Gamma^{\frac{p}{2}}\left(\frac{3}{p}\right)}{\Gamma^{\frac{p}{2}}\left(\frac{1}{p}\right)} \right) \tag{52}$$

respectively. Again, all these closed-form expressions for the outage probabilities are expressed in terms of simple functions that are easy to calculate. Note that, when a is an integer, the incomplete Gamma function $\gamma(a, x)$ can be expanded as a finite series [21, eq. (8.352)]

$$\gamma(a, x) = \Gamma(a) - \Gamma(a) e^{-x} \sum_{k=0}^{a-1} \frac{x^k}{k!}. \tag{53}$$

In this case, the calculations of the cdf and outage probability in the preceding discussion can be further simplified by using (53) in (28), (36), (42), and (50)–(52).

B. Rate Offset

The rate offset is defined as the difference between the ϵ-outage capacity and the capacity of an additive white Gaussian noise channel. In [3, eq. (6)], it was shown that the ϵ-outage capacity at large SNR is given by

$$C_{\epsilon}^{n}(SNR) = \log_2(SNR) + F_{\epsilon}^{-1} \left(\frac{1}{L} \sum_{i=1}^{L} \log_2\left(|h_{i,s}|^2\right) \right) + o(1) \tag{54}$$

where $\log_2(SNR)$ represents the capacity of an additive white Gaussian noise channel, $o(1)$ represents a term that goes to zero when the SNR goes to infinity, and $F_{\epsilon}^{-1}(X)$ denotes the solution y to the equation $Pr\{X < y\} = \epsilon$, $X = (1/L)\sum_{i=1}^{L} \log_2(|h_{i,s}|^2)$ in this case, L is the number of blocks, s indicates the name of the specific channel model, ϵ is the outage constraint, and $h_{i,s}$ is the channel gain for the ith block with normalized average power. Thus, the rate offset is given by $L_{\infty} = -F_{\epsilon}^{-1}((1/L)\sum_{i=1}^{L} \log_2(|h_{i,s}|^2))$. A closed-form expression for the rate offset at high SNRs was not found due to the difficulty in calculating the inverse function of the Meijer-G function. Using our new approximations, one has

$$\begin{aligned} \epsilon &= \Pr\left\{ \frac{1}{L} \sum_{i=1}^{L} \log_2\left(|h_{i,s}|^2\right) \le -L_{\infty} \right\} \\ &= \Pr\left\{ \prod_{i=1}^{L} |h_{i,s}|^2 \le 2^{-LL_{\infty}} \right\} \end{aligned} \tag{55}$$

where $h_{i,s} = (X'_{i,s}/\sqrt{E\{|X'_{i,s}|^2\}})$, such that $\prod_{i=1}^{L} |h_{i,s}|^2 = \tilde{\alpha}$ defined in (45). Therefore

$$\epsilon = \Pr\{\tilde{\alpha} \le 2^{-LL_{\infty}}\}. \tag{56}$$

Using the new approximations to the pdfs or the cdfs of the products of generalized Gamma, generalized Nakagami-m, and generalized Gaussian random variables, one has the rate offset for the generalized Gamma fading channel as

$$L_{\infty} = -\frac{2}{v \ln 2} \ln \left[\frac{\Omega_0}{2m_0} \frac{\Gamma^{\frac{v}{2}}(m)}{\Gamma^{\frac{v}{2}}\left(m + \frac{2}{v}\right)} \gamma^{-1}\left(m_0 + mL - L, \epsilon\right) \right] \tag{57}$$

for the generalized Nakagami-m fading channel as

$$L_{\infty} = -\frac{1}{v \ln 2} \ln \left[\frac{\Omega_0}{2m_0} \frac{\Gamma^{v}(m)}{\Gamma^{v}\left(m + \frac{1}{v}\right)} \gamma^{-1}\left(m_0 + mL - L, \epsilon\right) \right] \tag{58}$$

and for the generalized Gaussian fading channel as

$$L_{\infty} = -\frac{2}{p \ln 2} \ln \left[\frac{\Omega_0}{2m_0} \frac{\Gamma^{\frac{p}{2}}\left(\frac{1}{p}\right)}{\Gamma^{\frac{p}{2}}\left(\frac{3}{p}\right)} \gamma^{-1}\left(\frac{L}{p}, \epsilon\right) \right] \tag{59}$$

where $\gamma^{-1}(\cdot, \cdot)$ is the inverse function of the lower incomplete Gamma function, $m_0 = 0.6102 * L + 0.4263$, and $\Omega_0 = 0.8808 * L^{-0.9661} + 1.12$ in this case. The inverse incomplete Gamma function is much easier to calculate than the inverse Meijer-G function used in [3]. This shows the usefulness of our new approximations. Table I lists the derived approximate distribution functions for the generalized Gamma, Nakagami-m, and Gaussian RVs.

V. Numerical Results and Discussion

In this section, numerical results are presented to show the accuracy and the usefulness of the derived new approximations. In these examples, the approximate result is calculated using our new approximations, whereas the exact result is calculated using the Meijer-G function method.

Tables II–V compare the new approximate cdfs for the products of independent generalized Gamma, Weibull, generalized Nakagami-m, and generalized Gaussian RVs, respectively, with the exact cdfs from the Meijer G function method. The normalized RVs are used to simplify the comparison because the additional scale parameter will introduce more cases for comparison. One sees that the approximate cdfs agree quite well with the exact cdfs in most cases considered. For the generalized Gamma RV and the generalized Nakagami-m RV,

TABLE I
APPROXIMATE CDF FOR THE PRODUCT OF n INDEPENDENT RVs

GG (normalized)	$\gamma\left(m_0+mn-n,\, \frac{2m_0}{\Omega_0} y^{\frac{v}{n}}\right)$
GG	$\gamma\left(m_0+mn-n, m\frac{2m_0}{\Omega_0}\left(\frac{y}{\prod_{i=1}^{n}\Omega_i}\right)^{\frac{v}{n}}\right)$
WB (normalized)	$\gamma\left(m_0,\, \frac{2m_0}{\Omega_0} y^{\frac{v}{n}}\right)$
WB	$\gamma\left(m_0,\, \frac{2m_0}{\Omega_0}\left(\frac{y}{\prod_{i=1}^{n}\Omega_i}\right)^{\frac{v}{n}}\right)$
GN (normalized)	$\gamma\left(m_0+mn-n,\, \frac{2m_0}{\Omega_0} y^{\frac{2v}{n}}\right)$
GN	$\gamma\left(m_0+mn-n,\, \frac{2m_0}{\Omega_0}\left(\frac{\beta^{\frac{n}{2}} y}{\prod_{i=1}^{n}\Omega_i^{\frac{1}{2}}}\right)^{\frac{2v}{n}}\right)$
GA(normalized)	$0.5+0.5sign(y)\gamma\left(m_0-n+\frac{n}{p},\, \frac{m_0}{\Omega_0}\lvert y\rvert^{\frac{p}{n}}\right)$
GA	$0.5 \quad + \quad 0.5sign(y) \gamma\left(m_0-n+\frac{n}{p},\, \frac{m_0}{\Omega_0}\frac{2(\frac{\Gamma(\frac{3}{p})}{\Gamma(\frac{1}{p})})^{\frac{p}{2}}\lvert y\rvert^{\frac{p}{n}}}{(\prod_{i=1}^{n}\sigma_i)^{\frac{p}{n}}}\right)$

TABLE II
COMPARISON OF THE APPROXIMATE AND EXACT CDFs FOR INDEPENDENT NORMALIZED GENERALIZED GAMMA RVs

y	Exact $n=3$ $m=v=1$	Approximate $n=3$ $m=v=1$	Exact $n=6$ $m=v=2$	Approximate $n=6$ $m=v=2$
0.1	0.3590	0.3556	0.0009	0.0009
0.4	0.6109	0.6080	0.0201	0.0211
0.7	0.7150	0.7134	0.0570	0.0604
1.0	0.7764	0.7756	0.1027	0.1090
1.3	0.8176	0.8174	0.1520	0.1614
1.6	0.8473	0.8476	0.2019	0.2142
1.9	0.8698	0.8704	0.2508	0.2657
2.2	0.8874	0.8882	0.2979	0.3151
2.5	0.9015	0.9024	0.3425	0.3617
2.8	0.9131	0.9141	0.3846	0.4055
3.1	0.9227	0.9238	0.4241	0.4463
3.4	0.9307	0.9319	0.4610	0.4843
3.7	0.9376	0.9388	0.4954	0.5195
4.0	0.9435	0.9447	0.5274	0.5521
4.3	0.9487	0.9499	0.5572	0.5824
4.6	0.9531	0.9544	0.5824	0.6103
4.9	0.9571	0.9583	0.6107	0.6362

TABLE III
COMPARISON OF THE APPROXIMATE AND EXACT CDFs FOR INDEPENDENT NORMALIZED WEIBULL RVs

y	Exact $n=3$ $v=3$	Approximate $n=3$ $v=3$	Exact $n=6$ $v=5$	Approximate $n=6$ $v=5$
0.1	0.0220	0.0234	0.0123	0.0135
0.4	0.2911	0.2882	0.3351	0.3322
0.7	0.5815	0.5784	0.6865	0.6842
1.0	0.7764	0.7756	0.8729	0.8731
1.3	0.8872	0.8880	0.9521	0.9530
1.6	0.9453	0.9465	0.9826	0.9834
1.9	0.9741	0.9752	0.9938	0.9943
2.2	0.9880	0.9888	0.9978	0.9980
2.5	0.9945	0.9950	0.9992	0.9993
2.8	0.9975	0.9978	0.9997	0.9998
3.1	0.9989	0.9990	0.9999	0.9999
3.4	0.9995	0.9996	1.0000	1.0000
3.7	0.9998	0.9998	1.0000	1.0000
4.0	0.9999	0.9999	1.0000	1.0000
4.3	1.0000	1.0000	1.0000	1.0000
4.6	1.0000	1.0000	1.0000	1.0000
4.9	1.0000	1.0000	1.0000	1.0000

TABLE IV
COMPARISON OF THE APPROXIMATE AND EXACT CDFs FOR INDEPENDENT NORMALIZED GENERALIZED NAKAGAMI-m RVs

y	Exact $n=3$ $m=v=1$	Approximate $n=3$ $m=v=1$	Exact $n=6$ $m=v=2$	Approximate $n=6$ $m=v=2$
0.1	0.1035	0.1038	0.0000	0.0000
0.4	0.4392	0.4355	0.0028	0.0029
0.7	0.6493	0.6469	0.0298	0.0314
1.0	0.7764	0.7756	0.1027	0.1090
1.3	0.8547	0.8550	0.2167	0.2298
1.6	0.9040	0.9050	0.3511	0.3707
1.9	0.9357	0.9368	0.4853	0.5092
2.2	0.9563	0.9575	0.6057	0.6312
2.5	0.9700	0.9711	0.7061	0.7309
2.8	0.9792	0.9802	0.7857	0.8082
3.1	0.9855	0.9863	0.8464	0.8658
3.4	0.9898	0.9904	0.8914	0.9074
3.7	0.9927	0.9933	0.9241	0.9369
4.0	0.9948	0.9953	0.9474	0.9574
4.3	0.9963	0.9966	0.9638	0.9714
4.6	0.9973	0.9976	0.9752	0.9810
4.9	0.9980	0.9983	0.9831	0.9874

the difference between the approximate cdf and the exact cdf only appears in the third digit after the decimal point for most values of y examined when $m=v=1$ and $n=3$. When $m=v=2$ and $n=6$, the difference between the approximate cdf and the exact cdf is noticeable. For the Weibull RV, there is an excellent agreement for all values of y. For the generalized Gaussian RV, the difference between the approximate cdf and the exact cdf is small when the value of y is large. When the value of y is small, there is considerable disagreement between the approximate cdf and the exact cdf. Table VI shows the absolute relative errors of the approximate cdfs, which are defined as $|F1-F2|/F1$, where $F1$ represents the exact cdf and $F2$ represents the corresponding approximate cdf, to compare the accuracy of the approximation from the analysis. This also indicates the relative distance between the exact cdf and the approximate cdf. One sees that the largest error for the generalized Gamma RV is about 0.0616, the largest error for the Weibull RV is about 0.0966, the largest error for the generalized Nakagami-m RV is about 0.0615, and the largest error for the generalized Gaussian is about 0.0339. Thus, the upper bound of the approximation error is 0.062 in the cases considered. Overall, the approximation accuracy is high. Fig. 1 compares the absolute relative errors for different numbers of RVs in (23), which has been used to derive the approximations for the generalized Gamma, Nakagami-m, and Gaussian RVs. One sees that the absolute relative error is less than 0.015 in all the cases considered. This verifies the accuracy and validity of the proposed approximation in (23). One also sees that, when $1 < y < 2.5$, the absolute relative error decreases when n increases, whereas when $4.5 < y < 5$, the absolute relative error increases when n increases. Therefore, the accuracy of the approximation may increase or decrease when n increases, depending on the value of y.

Figs. 2–4 show the outage probability performance of the wireless relaying system in cascaded generalized Gamma, generalized Nakagami-m, and generalized Gaussian fading

TABLE V
COMPARISON OF THE APPROXIMATE AND EXACT CDFS FOR INDEPENDENT NORMALIZED GENERALIZED GAUSSIAN RVS

y	Exact $n=3$ $p=1$	Approximate $n=3$ $p=1$	Exact $n=6$ $p=2$	Approximate $n=6$ $p=2$
0.1	0.5593	0.5593	0.8441	0.8727
0.4	0.6287	0.6274	0.9335	0.9453
0.7	0.6689	0.6673	0.9587	0.9659
1.0	0.6980	0.6963	0.9710	0.9760
1.3	0.7210	0.7191	0.9783	0.9820
1.6	0.7399	0.7380	0.9831	0.9859
1.9	0.7559	0.7541	0.9864	0.9886
2.2	0.7697	0.7680	0.9888	0.9907
2.5	0.7819	0.7802	0.9906	0.9922
2.8	0.7927	0.7911	0.9921	0.9934
3.1	0.8024	0.8009	0.9932	0.9943
3.4	0.8112	0.8098	0.9941	0.9951
3.7	0.8192	0.8179	0.9949	0.9957
4.0	0.8266	0.8253	0.9955	0.9962
4.3	0.8333	0.8322	0.9960	0.9967
4.6	0.8396	0.8385	0.9964	0.9970
4.9	0.8454	0.8444	0.9968	0.9973

TABLE VI
ABSOLUTE RELATIVE ERRORS FOR THE APPROXIMATE CDFS

y	Generalized Gamma $n=6$ $m=v=2$	Weibull $n=6$ $v=5$	Generalized Nakagami $n=6$ $m=v=2$	Generalized Gaussian $n=6$ $p=2$
0.1	0.0352	0.0966	0.0506	0.0339
0.4	0.0537	0.0088	0.0409	0.0127
0.7	0.0596	0.0034	0.0562	0.0075
1.0	0.0615	0.0002	0.0615	0.0051
1.3	0.0616	0.0010	0.0604	0.0037
1.6	0.0607	0.0008	0.0557	0.0029
1.9	0.0594	0.0005	0.0493	0.0023
2.2	0.0578	0.0002	0.0422	0.0019
2.5	0.0561	0.0001	0.0352	0.0016
2.8	0.0542	0.0000	0.0287	0.0013
3.1	0.0524	0.0000	0.0229	0.0011
3.4	0.0505	0.0000	0.0180	0.0010
3.7	0.0487	0.0000	0.0139	0.0008
4.0	0.0469	0.0000	0.0106	0.0007
4.3	0.0452	0.0000	0.0079	0.0007
4.6	0.0435	0.0000	0.0059	0.0006
4.9	0.0419	0.0000	0.0043	0.0005

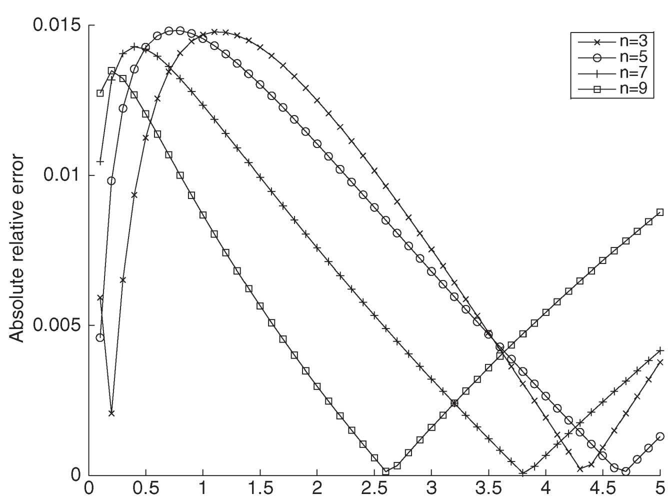


Fig. 1. Absolute relative error for different numbers of independent normalized exponential RVs.

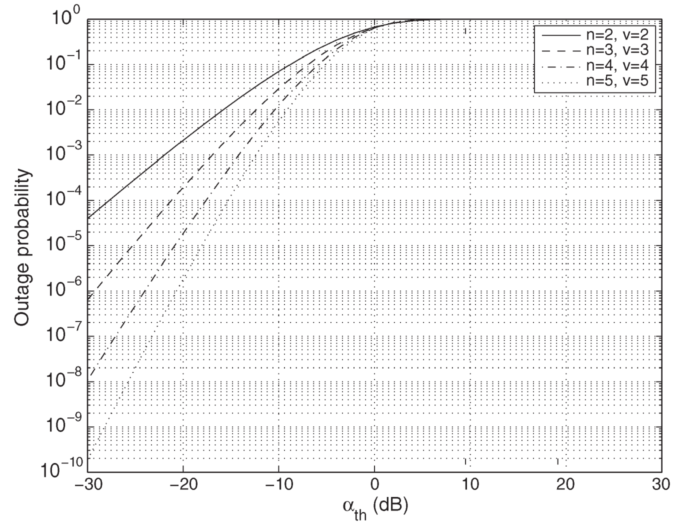


Fig. 2. Outage probability for different distribution parameters and different numbers of independent RVs in cascaded generalized Gamma fading channels when $m = 2$.

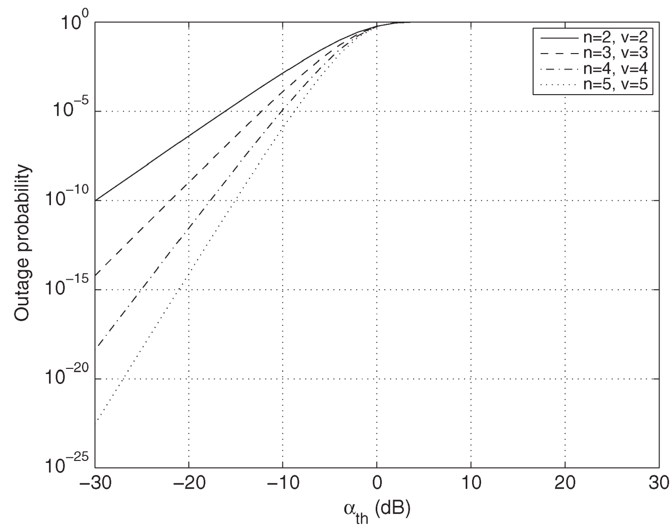


Fig. 3. Outage probability for different distribution parameters and different numbers of independent RVs in cascaded generalized Nakagami-m fading channels when $m = 2$.

channels, respectively. As expected, the outage probability increases when the SNR threshold increases. In the extreme case when the SNR threshold goes to infinity, the outage probability will approach 1. In general, the generalized Gaussian channel gives the largest outage probability, whereas the generalized Nakagami-m channel gives the smallest outage probability, which implies that the generalized Gaussian channel has the toughest channel condition for signal transmission. However, a more meaningful comparison should be made based on their specific parameters. Moreover, the outage probabilities for the generalized Gamma and generalized Nakagami-m increase much faster than the outage probability for the generalized Gaussian, when the SNR threshold increases. This can be explained from (50)–(52). Recall that the lower incomplete Gamma function is defined in (26). By taking differentiations of P_o with respect to $\alpha_{\rm th}$, one can easily find that it is the parameter of a in the lower incomplete Gamma function that determines the slope of the curve when $p = v$. Since

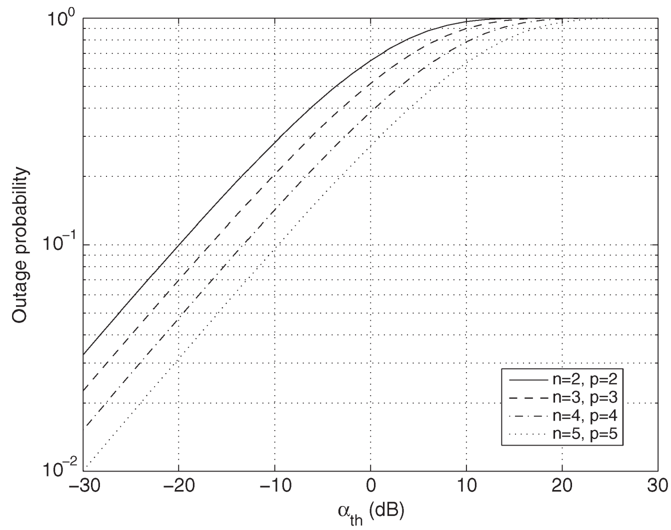


Fig. 4. Outage probability for different distribution parameters and different numbers of independent RVs in cascaded generalized Gaussian fading channels.

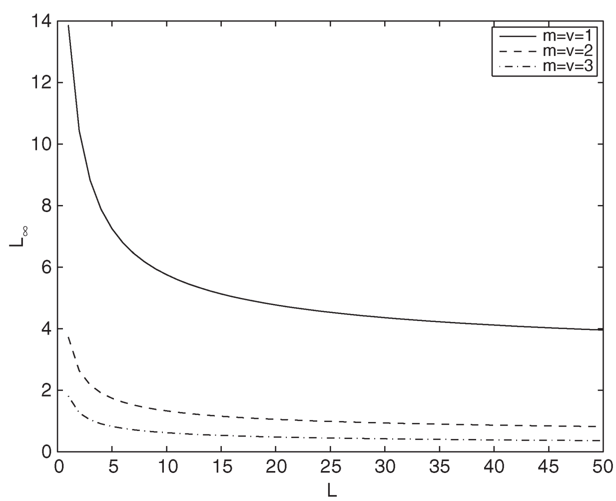


Fig. 5. Rate offset of H-ARQ for different distribution parameters in the generalized Gamma fading channels at high SNRs.

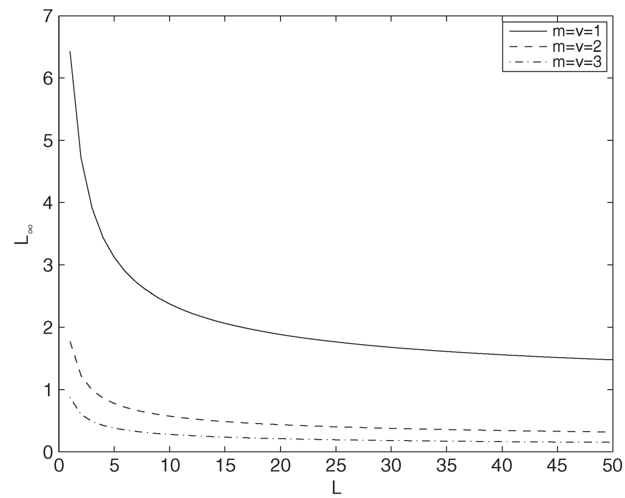


Fig. 6. Rate offset of H-ARQ for different distribution parameters in the generalized Nakagami-m fading channels at high SNRs.

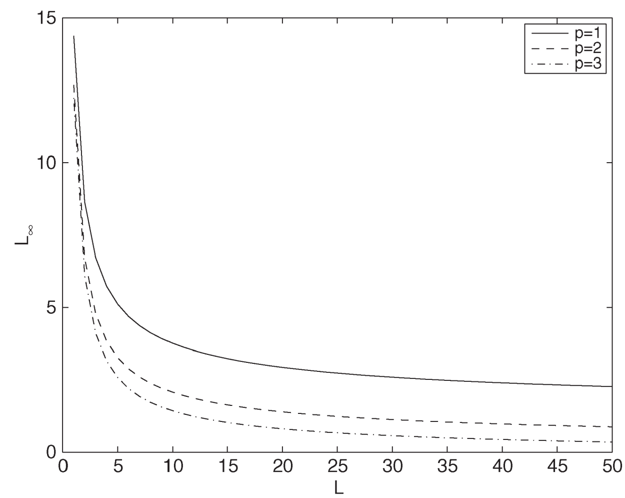


Fig. 7. Rate offset of H-ARQ for different distribution parameters in the generalized Gaussian fading channels at high SNRs.

$a = (n/p)$ for the generalized Gaussian channel is smaller than $a = m_0 + mn - n$ for the generalized Gamma and the generalized Nakagami-m channels in the cases considered, one has a smaller slope for the generalized Gaussian channel.

Figs. 5–7 show the rate offset of the H-ARQ transmission in the generalized Gamma, generalized Nakagami-m, and generalized Gaussian fading channels, respectively. The value of ϵ is set to 0.01. For the generalized Gamma and generalized Nakagami-m fading channels, the rate offset decreases with the number of blocks, and they approach a lower limit. This agrees with that observed in [3], which considered a Rayleigh fading channel, as expected, as Rayleigh is a special case of the generalized Gamma or generalized Nakagami-m fading channels. The rate offset also decreases when the parameters of m and v increase, suggesting that one has different capacity performances for different channel conditions. However, the amount of decrease reduces when the parameters of m and v increase. Therefore, in addition to the lower limit of the rate offset with respect to the number of blocks, there is also a lower limit of the rate offset with respect to the values of m and v. For the generalized Gaussian fading channel, one sees that the rate offset very quickly decreases at small values of block number. In addition, at small values of block numbers, the rate offset does not monotonically decrease when the value of p increases. This is due to the additional dependence of $\gamma^{-1}((L/p), \epsilon)$ on the value of p in (59), as other channels do not have similar dependence on the value of v in their expressions; therefore, their rate offset does monotonically decrease when the value of v increases. Due to this difference, L_∞ in Figs. 5 and 6 changes in a different way from L_∞ in Fig. 7 with respect to the model parameters.

VI. Conclusion

New approximate pdfs and cdfs for the products of independent generalized Gamma, generalized Nakagami-m, and generalized Gaussian RVs have been derived. Calculations of the pdfs and cdfs of the products of independent RVs based on these new approximations are much simpler than those using the existing methods. Numerical results have shown that the new approximations also have satisfactory accuracies. Using these new approximations, new closed-form expressions for the outage probability in the wireless cascaded channel and the rate offset for H-ARQ transmission have been obtained and examined. The outage probability behavior and the rate offset behavior have also been explained using the derived expressions based on the new approximate pdfs and cdfs, which is otherwise not possible using expressions based on the existing methods. Examples for the MIMO keyhole systems can be examined in a similar manner. Due to the length restriction, they are not shown here.

Appendix
Derivations of (24), (33), and (39)

From (9), the Mellin transform of a pdf is actually equal to the $(s-1)$th-order moment of the RV. Using [17, eq. (3)], the Mellin transform of the generalized Gamma pdf is

$$M_{X_{i,GG}}(s) = \frac{\Gamma\left(m + \frac{s-1}{v}\right)}{\Gamma(m)}. \quad (60)$$

Thus, the Mellin transform of the product of n independent normalized generalized Gamma RVs can be derived from (60) as

$$M_{Y_{GG}}(s) = \frac{\Gamma^n\left(m + \frac{s-1}{v}\right)}{\Gamma^n(m)}. \quad (61)$$

Then, the exact pdf of the product of n independent normalized generalized Gamma RVs can be calculated as the inverse Mellin transform

$$f_{Y_{GG}}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\frac{\Gamma^n\left(m + \frac{s-1}{v}\right)}{\Gamma^n(m)}ds. \quad (62)$$

Let $s' = m + (s - 1/v)$. One can simplify (62) as

$$f_{Y_{GG}}(x) = \frac{vx^{mv-1}}{\Gamma^n(m)} \cdot \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} (x^v)^{-s'}\Gamma^n(s')ds'. \quad (63)$$

Finally, by using (23) in (63), taking an integration of the resulting function over x from 0 to ∞, and normalizing the function with respect to the integral, (24) is derived. In the preceding result, normalization with respect to the integral of the resulting function from 0 to ∞ takes the approximation error into account, so that the overall area under the pdf is equal to 1.

Next, consider the generalized Nakagami-m distribution. Using the moment expression in [11], one has the Mellin transform of the generalized Nakagami-m pdf as

$$M_{X_{i,GN}}(s) = \frac{\Gamma\left(m + \frac{s-1}{2v}\right)}{\Gamma(m)} \quad (64)$$

which gives the Mellin transform of the product of n independent normalized generalized Nakagami-m RVs as

$$M_{Y_{GN}}(s) = \frac{\Gamma^n\left(m + \frac{s-1}{2v}\right)}{\Gamma^n(m)}. \quad (65)$$

Similarly, using (65), the exact pdf of the product of n independent normalized generalized Nakagami-m RVs is calculated as the inverse Mellin transform

$$f_{Y_{GN}}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\frac{\Gamma^n\left(m + \frac{s-1}{2v}\right)}{\Gamma^n(m)}ds. \quad (66)$$

After variable transformation, substitution of (23) and normalization with respect to the integral, so that the overall area under the pdf curve equals 1, (33) can be derived. Note that (33) can also be directly derived from (24) by conducting a variable transformation.

In the case of the generalized Gaussian distribution, the method is slightly different from the preceding discussion because the generalized Gaussian RV takes both positive and negative values. In this case, as shown in [6] and [19], this difficulty can be overcome by splitting the positive and negative components of the generalized Gaussian pdf to define two functions as

$$f_{X^+_{i,GA}}(x) = \frac{p}{2^{1+\frac{1}{p}}\Gamma\left(\frac{1}{p}\right)}e^{-\frac{1}{2}|x|^p}S(x) \quad (67)$$

$$f_{X^-_{i,GA}}(x) = \frac{p}{2^{1+\frac{1}{p}}\Gamma\left(\frac{1}{p}\right)}e^{-\frac{1}{2}|x|^p}S(-x) \quad (68)$$

where $S(x)$ is the step function with $S(x) = 1$ for $x > 0$ and $S(x) = 0$ otherwise. From [20], the Mellin transform of $f_{X^+_{i,GA}}(x)$ is given by

$$M_{X^+_{i,GA}}(s) = \frac{2^{\frac{s-1}{p}-1}}{\Gamma\left(\frac{1}{p}\right)}\Gamma\left(\frac{s}{p}\right). \quad (69)$$

Then, using [6, eq. (18)], the Mellin transforms of the positive component and the negative component of the product of n independent normalized generalized Gaussian RVs are given by

$$M_{Y^+_{GA}}(s) = M_{Y^-_{GA}}(s) = 2^{n-1}\frac{2^{n\frac{s-1}{p}-n}}{\Gamma^n\left(\frac{1}{p}\right)}\Gamma^n\left(\frac{s}{p}\right) \quad (70)$$

where the coefficient of 2^{n-1} takes into account the fact that $f_{X^+_{i,GA}}(x)$ only describes half of the possible values for the generalized Gaussian RV.

From (70), the exact pdf of the positive component of the product of independent normalized generalized Gaussian RVs is derived as

$$f_{Y^+_{GA}}(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}2^{n-1}\frac{2^{n\frac{s-1}{p}-n}}{\Gamma^n\left(\frac{1}{p}\right)}\Gamma^n\left(\frac{s}{p}\right)ds. \quad (71)$$

By taking a variable transformation of $s' = (s/p)$, (71) can be simplified as

$$f_{Y^+_{GA}}(x) = \frac{2^{-\frac{n}{p}-1}p}{\Gamma^n\left(\frac{1}{p}\right)} \cdot \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{y^p}{2^n}\right)^{-s'}\Gamma^n(s')ds'. \quad (72)$$

Using (23) in (72), taking an integration of the resulting function over x from 0 to ∞ and normalizing the resulting function with respect to the integral multiplied by 0.5, as $f_{Y_{GA}^{+}}(x)$ only defines the pdf in half of the plane, one has the approximate pdf of the positive component of the product of independent normalized generalized Gaussian RVs as

$$f_{Y_{GA}^{+}}(x) \approx \frac{p}{2n\Gamma\left(m_0 - n + \frac{n}{p}\right)} \left(\frac{m_0}{\Omega_0}\right)^{m_0 - n + \frac{n}{p}} \times x^{\frac{pm_0}{n} - p} e^{-\frac{m_0}{\Omega_0} x^{\frac{p}{n}}}, \qquad x > 0. \quad (73)$$

Similarly, the approximate pdf of the negative component of the product of independent normalized generalized Gaussian RVs is

$$f_{Y_{GA}^{-}}(-x) \approx \frac{p}{2n\Gamma\left(m_0 - n + \frac{n}{p}\right)} \left(\frac{m_0}{\Omega_0}\right)^{m_0 - n + \frac{n}{p}} \times (-x)^{\frac{pm_0}{n} - p} e^{-\frac{m_0}{\Omega_0} (-x)^{\frac{p}{n}}}, \qquad x \le 0. \quad (74)$$

Combining the two components, (39) can be derived.

REFERENCES

[1] J. N. Laneman and G. W. Wornell, "Energy-efficient antenna sharing and relaying for wireless networks," in *Proc. IEEE Wireless Commun. Netw. Conf.*, Chicago, IL, Mar. 2000, vol. 1, pp. 7–12.

[2] J. Salo, H. M. El-Sallabi, and P. Vainikainen, "Statistical analysis of the multiple scattering radio channel," *IEEE Trans. Antennas Propag.*, vol. 54, no. 11, pp. 3114–3124, Nov. 2006.

[3] P. Wu and N. Jindal, "Performance of hybrid-ARQ in block-fading channels: A fixed outage probability analysis," *IEEE Trans. Commun.*, vol. 58, no. 4, pp. 1129–1141, Apr. 2010.

[4] P. Hesami and J. N. Laneman, "Incremental use of multiple transmitters for low-complexity diversity transmission in wireless systems," *IEEE Trans. Commun.*, submitted for publication. [Online]. Available: http://www.nd.edu/~jnl/pubs/tcomm2011b.pdf

[5] G. K. Karagiannidis, N. C. Sagias, and P. T. Mathiopoulos, "N *Nakagami: a novel stochastic model for cascaded fading channels," *IEEE Trans. Commun.*, vol. 55, no. 8, pp. 1453–1458, Aug. 2007.

[6] M. D. Springer and W. E. Thompson, "The distribution of products of independent random variables," *SIAM J. Appl. Math.*, vol. 14, no. 3, pp. 511–526, May 1966.

[7] M. D. Springer and W. E. Thompson, "The distribution of products of Beta, Gamma and Gaussian random variables," *SIAM J. Appl. Math.*, vol. 18, no. 4, pp. 721–737, Jun. 1970.

[8] B. D. Carter and M. D. Springer, "The distribution of products, quotients and powers of independent H-function variates," *SIAM J. Appl. Math.*, vol. 33, no. 4, pp. 542–558, Dec. 1977.

[9] J. Salo, H. M. El-Sallabi, and P. Vainikainen, "The distribution of the product of independent Rayleigh random variables," *IEEE Trans. Antennas Propag.*, vol. 54, no. 2, pp. 639–643, Feb. 2006.

[10] S. Ahmed, L.-L. Yang, and L. Hanzo, "Probability distributions of products of Rayleigh and Nakagami-m variables using Mellin transform," in *Proc. IEEE ICC*, Kyoto, Japan, Jun. 2011, pp. 1–5.

[11] F. Yilmaz and M.-S. Alouini, "Product of the powers of generalized Nakagami-m variates and performance of cascaded fading channels," in *Proc. Globecom*, Honolulu, HI, Dec. 2009, pp. 1–8.

[12] M. D. Yacoub, "The $\alpha - \mu$ distribution: A physical fading model for the Stacy distribution," *IEEE Trans. Veh. Technol.*, vol. 56, no. 1, pp. 27–34, Jan. 2007.

[13] E. W. Stacy, "A generalization of the gamma distribution," *Ann. Math. Stat.*, vol. 33, no. 3, pp. 1187–1192, Sep. 1962.

[14] N. C. Beaulieu and D. J. Young, "Designing time-hopping ultrawide bandwidth receivers for multiuser interference environments," *Proc. IEEE*, vol. 97, no. 2, pp. 255–284, Feb. 2009.

[15] Y. Chen and N. C. Beaulieu, "Novel low-complexity estimators for the shape parameter of the generalized Gaussian distribution," *IEEE Trans. Veh. Technol.*, vol. 58, no. 4, pp. 2067–2071, May 2009.

[16] H. Lu, Y. Chen, and N. Cao, "Accurate approximation to the PDF of the product of independent Rayleigh random variables," *IEEE Antennas Wireless Propag. Lett.*, vol. 10, pp. 1019–1022, Oct. 2011.

[17] E. W. Stacy and G. A. Mihram, "Parameter estimation for a generalized Gamma distribution," *Technometrics*, vol. 7, no. 3, pp. 349–358, Aug. 1965.

[18] M. K. Simon and M.-S. Alouini, *Digital Communication Over Fading Channels*, 2nd ed. Hoboken, NJ: Wiley, 2005.

[19] B. Epstein, "Some applications of the Mellin transform in statistics," *Ann. Math. Stat.*, vol. 19, no. 3, pp. 370–379, Sep. 1948.

[20] S. Nadarajah, "A generalized normal distribution," *J. Appl. Stat.*, vol. 32, no. 7, pp. 685–694, Sep. 2005.

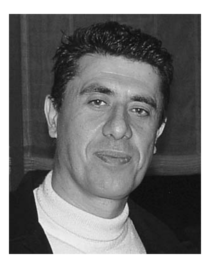
[21] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. New York: Academic, 2000.

[22] D. S. Michalopoulos and T. A. Tsiftsis, "Performance analysis of wireless multihop diversity systems," *Int. J. Commun. Syst.*, vol. 21, no. 9, pp. 955–969, Sep. 2008.

[23] T. A. Tsiftsis, "Performance of wireless multihop communications systems with cooperative diversity over fading channels," *Int. J. Commun. Syst.*, vol. 21, no. 5, pp. 559–565, May 2008.

Yunfei Chen (S'02–M'06–SM'10) received the B.E. and M.E. degrees in electronics engineering from Shanghai Jiaotong University, Shanghai, China, in 1998 and 2001, respectively, and the Ph.D. degree from the University of Alberta, Edmonton, AB, Canada, in 2006.

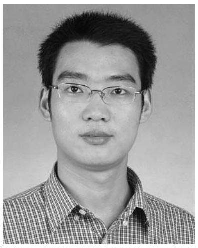
He is currently an Associate Professor with the School of Engineering, University of Warwick, Coventry, U.K. He is also with the School of Computer and Information, Hohai University, Nanjing, China. His research interests include wireless communications theory, cognitive radio, and ultrawideband systems.



George K. Karagiannidis (M'97–SM'04) was born in Pithagorion, Samos Island, Greece. He received the University Diploma (5 years) and Ph.D degree, both in electrical and computer engineering, from the University of Patras, Greece, in 1987 and 1999, respectively. From 2000 to 2004, he was a Senior Researcher with the Institute for Space Applications and Remote Sensing, National Observatory of Athens, Greece. In June 2004, he joined the faculty of Aristotle University of Thessaloniki, Greece, where he is currently Associate Professor (four-level academic rank system) of Digital Communications Systems with the Electrical and Computer Engineering Department and Head of the Telecommunications Systems and Networks Lab. His research interests are in the broad area of digital communications systems with emphasis on wireless communications theory, energy efficient MIMO and cooperative communications, cognitive radio, wireless security, and optical wireless communications.

He is the author or co-author of more than 150 technical papers published in scientific journals and presented at international conferences. He is also a co-author of three chapters in books and author of the Greek edition of a book on Telecommunications Systems. He is co-recipient of the Best Paper Award from the Wireless Communications Symposium (WCS) at the IEEE International Conference on Communications (ICC'07), Glasgow, U.K., June 2007. He has been a member of Technical Program Committees for several IEEE conferences such as ICC, GLOBECOM, VTC, etc. In the past, he has been Editor for Fading Channels and Diversity of the IEEE TRANSACTIONS ON COMMUNICATIONS, Senior Editor of IEEE COMMUNICATIONS LETTERS, and Editor of the *EURASIP Journal of Wireless Communications & Networks*. He was Lead Guest Editor of the special issue on "Optical Wireless Communications" of the IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS and currently serves as a Guest Editor of the special issue on "Large-scale multiple antenna wireless systems" (to be published in February 2013).

Dr. Karagiannidis has been the Editor-in Chief of the IEEE COMMUNICATIONS LETTERS since January 2012.



Hao Lu received the B.S. and M.S. degrees in electrical engineering, in 2003 and 2007, respectively, from Hohai University, Nanjing, China, where he is currently working toward the Ph.D. degree with the College of Energy and Electric.

His research interests include statistical signal processing, sensor array processing, and sparse signal representations.

Ning Cao received the B.S. and M.S. degrees in electrical engineering from Southeast University, Nanjing, China, in 1984 and 1990, respectively.

From 1991 to 2002, he was an Assistant Professor with the College of Computer and Information, Hohai University, Changzhou, China. Since May 2002, he has been with the School of Computer and Information, Hohai University, Nanjing, where he is currently a Professor. His current research interests include statistical signal processing, sensor array processing, and wireless communications.