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# Block error rate of optical wireless communication systems over atmospheric turbulence channels

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**Abstract:** Block error rate performance of subcarrier intensity modulation based optical wireless communication systems is analysed over Gamma–Gamma and lognormal atmospheric turbulence channels. The analysis is applicable to non-coherent binary modulations and coherent binary phase shift keying. The special cases of $K$-distributed strong turbulence channel and negative exponential turbulence channel are also considered. The closed-form expressions of block error rate in the Gamma–Gamma turbulence channels are derived using a series expansion of the modified Bessel function. For the lognormal turbulence channels, the Gauss–Laguerre quadrature method can be used to estimate the block error rate accurately.

## 1 Introduction

Atmospheric turbulence induced fading is the main impairment on the performance of terrestrial optical wireless communication (OWC) systems [1]. Fading or scintillation is caused by variations in the refractive index because of inhomogeneities in temperature and pressure fluctuations. The optical intensity fluctuations at the receiver can severely degrade the link performance, especially for link distances over 1 km or longer [2].

In OWC, the turbulence fading channel typically changes slowly. Despite the received optical intensity suffers random fluctuation, the coherence time of atmospheric turbulence channels is typically on the order of 1 ms, whereas the data rate of OWC systems can be on the order of Gbps [3]. As a result, the same channel fading coefficient will affect a large block of data bits. Therefore the bit-errors will have dependence, that is, the slow turbulence fading channel has memory. The traditional bit-error rate (BER) is inadequate to assess the performance of such systems, and block error rate (BLER) is a more meaningful performance metric.

For a non-fading channel, the bit errors are independently and identically distributed, and the errors in a block of $N$ bits are binomially distributed. We denote the probability of BLER or simply block error by $P(M, N)$, and it is defined as the probability of having more than $M$ bit errors within a block of $N$ bits. Thus, $P(M, N)$ is calculated by

$$P(M,\ N) = \sum_{m=M+1}^{N} \binom{N}{m} p^{m}(1-p)^{N-m} \qquad (1)$$

where $p$ is the probability of bit error.

BLER has several important applications. For examples, if a simple automatic repeat request system is used, the system performance is determined by the probability of occurrence of one or more bit errors in a block, that is, $P(0, N)$. On the other hand, if an error-correction code is to be employed to correct up to $M$ errors in each block of $N$ bits, the system performance is governed by $P(M, N)$. In the special case when we set $M = 0$ and $N = 1$, BLER becomes the BER. Therefore BLER is a generalised performance metric that is more suitable for slowly changing turbulence-induced fading channels.

In radio frequency (RF) wireless communication literature, BLER was studied for slow Rayleigh fading [4], and for slow Rayleigh fading with diversity reception [5]. Both works focused on non-coherent binary signallings, and did not treat the BLER of binary phase shift keying. In OWC literature, there are extensive works focused on the BER of OWC systems over atmospheric fading channels. In [6], the BER performance of a subcarrier intensity modulated (SIM) OWC system with binary phase shift keying (BPSK), differential phase shift keying (DPSK) and non-coherent frequency shift keying (NCFSK) was investigated over Gamma–Gamma turbulence channels. In [7], error rate performance of SIM OWC system was analysed over strong atmospheric turbulence channels. Various modulation schemes such as $M$-ary phase shift keying (MPSK), DPSK and NCFSK were studied. To our best knowledge, BLER performance has not been studied for OWC systems in slow atmospheric turbulence channels, which is a more accurate measurement of the error performance of such systems.

In this paper, we will analyse the BLER performance of SIM-based OWC systems employing both non-coherent and coherent binary modulations over various atmospheric turbulence models. We obtain highly accurate BLER expressions in terms of an infinite series over the Gamma–Gamma turbulence channels for non-coherent binary modulations. For coherent BPSK, we first propose a new


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sum of exponentials approximation of the Gaussian $Q$-function, and then develop the corresponding BLER expression. For the lognormal turbulence model, we use the Gauss–Laguerre quadrature method to obtain an accurate estimation of the BLER.

The remainder of this paper is organised as follows. In Section 2, we introduce the SIM system. Section 3 reviews some important atmospheric turbulence channel models. In Sections 4 and 5, we analyse the BLER performance of SIM systems employing both coherent and non-coherent binary modulations, respectively. Finally, Section 6 presents some numerical results, and Section 7 makes our conclusions.

## 2 Subcarrier intensity modulation

In a SIM system, a RF signal $s(t)$, which is pre-modulated with the data source and properly biased, is used to modulate the irradiance of a continuous wave optical beam at the laser transmitter. We normalise the power of $s(t)$ to be unity. At the receiver, the photodetector converts the received optical intensity to an electrical signal through direct detection. The photocurrent at the output of photodetector can be written as [7]

$$i(t) = RI(t)A[1 + \xi s(t)] + n(t) \tag{2}$$

where $R$ is the photodetector responsivity, $I(t)$ is assumed to be a stationary random process describing the irradiance fluctuation caused by the atmospheric turbulence, $A$ is the photodetector area, $n(t)$ is an additive white Gaussian noise process because of thermal or/and background noise/s and $\xi$ is the modulation index satisfying $-1 < \xi s(t) < 1$ in order to avoid overmodulation. The sample $I(t)$ at the time instant $t = t_0$ is a random variable (RV) $I$. The instantaneous signal-to-noise ratio (SNR) at the input of the electrical demodulator is given by [7, eq. (4)]

$$\gamma = \frac{(RA\xi)^2}{2\Delta f(qRI_b + 2k_b T_k/R_L)} I^2 = CI^2 \tag{3}$$

where $I_b$ is the background light irradiance, $q$ is the electronic charge, $\Delta f$ is the noise equivalent bandwidth of the photodetector, $k_b$ is the Boltzmann's constant, $T_k$ is the temperature in kelvin and $R_L$ is the load resistance. It can be shown that under the assumption of normalised mean irradiance, that is, $E[I] = 1$ where $E[\cdot]$ denotes expectation operation, the parameter $C$ is also the average electrical SNR $\bar{\gamma}$. Therefore in the ensuing analysis we will denote the SNR by $\gamma = \bar{\gamma} I^2$.

## 3 Atmospheric turbulence models

For outdoor OWC, several statistical models have been proposed to describe the irradiance fluctuation. The lognormal distribution is widely used to describe the weak turbulence channels over several 100 m [8]. The Gamma–Gamma distribution has recently emerged as a useful turbulence model because it provides a good fit to the experimental measurements of irradiance for both weak and strong turbulence channels [9]. As the special cases of the Gamma–Gamma model, the $K$-distribution is commonly used to model the irradiance in strong turbulence regimes with link distance approximately 1 km [10], and the negative exponential distribution can be used to describe the saturation regimes [11].

### 3.1 Lognormal model

For a lognormal turbulence channel, the optical irradiance $I$ is given by $I = \exp(-X)$, where $X$ is a Gaussian RV with mean $\mu$ and variance $\sigma^2$. In OWC, the scintillation index $\sigma$ has typical values between 0.02 and 0.5 [12], and it can never be greater than 0.75 [13]. Using the lognormal probability density function (PDF) in [14], we obtained the PDF of $I$ with a normalised mean as

$$f_L(I) = \frac{1}{\sqrt{2\pi}\sigma I} \exp\left[-\frac{(\ln I + \sigma^2/2)^2}{2\sigma^2}\right], \quad I > 0 \tag{4}$$

### 3.2 Gamma–Gamma model

When the optical irradiance $I$ is modelled as a Gamma–Gamma RV, it has the PDF (after normalising the mean) [6]

$$f_{GG}(I) = \frac{2}{\Gamma(\alpha)\Gamma(\beta)} (\alpha\beta)^{(\alpha+\beta/2)} I^{(\alpha+\beta/2)-1} K_{\alpha-\beta}(2\sqrt{\alpha\beta I}), \quad I > 0 \tag{5}$$

and where $\Gamma(\cdot)$ denotes the Gamma function and $K_{\alpha-\beta}(\cdot)$ is the modified Bessel function of the second kind of order $\alpha - \beta$. In OWC, the positive parameters $\alpha$ and $\beta$ are, respectively, the effective number of large-scale and small-scale cells of the scattering environment, and their values are determined by the Rytov variance [2]. The inequality $\alpha > \beta$ is valid for OWC applications under the assumption of plane wave and negligible inner scale [15]. Without loss of generality, this inequality condition is assumed to be true in this paper. The PDF in (5) can be alternatively expressed in series with the aid of a series expansion of the modified Bessel function [16, eq. (6)] as

$$f_{GG}(I) = \sum_{p=0}^{\infty} \left[a_p(\alpha, \beta) I^{p+\beta-1} + a_p(\beta, \alpha) I^{p+\alpha-1}\right], \quad I > 0 \tag{6}$$

where

$$a_p(x, y) = \frac{(xy)^{p+y}\Gamma(x-y)\Gamma(1-x+y)}{\Gamma(x)\Gamma(y)\Gamma(p-x+y+1)p!} \tag{7}$$

where we used the Euler reflective identity $\pi/\sin(\pi(x-y)) = \Gamma(x-y)\Gamma(1-x+y)$ [17, eq. 8.334 (3)].

### 3.3 K-distributed model

The $K$-distributed turbulence channel is another useful turbulence model describing irradiance fluctuations in strong turbulence conditions [2, 18, 19]. It is also a special case of the Gamma–Gamma model when $\beta = 1$. The PDF of a $K$-distributed RV $I$ is [7, eq. (8)]

$$f_K(I) = \frac{2}{\Gamma(\alpha)} \alpha^{(\alpha+1/2)} I^{(\alpha-1/2)} K_{\alpha-1}\left(2\sqrt{\alpha I}\right), \quad I > 0 \tag{8}$$

Alternatively, it can be obtained from (6) as

$$f_K(I) = \sum_{p=0}^{\infty} \left[a_p(\alpha, 1) I^p + a_p(1, \alpha) I^{p+\alpha-1}\right], \quad I > 0 \tag{9}$$

The typical values of the parameter $\alpha$ lie within (1, 2) [20].


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### 3.4 Negative exponential model

The negative exponential atmospheric turbulence channel is used to describe the limiting case of saturated scintillation [9, 21] with the corresponding PDF given by

$$f_{\rm NE}(I) = \exp(-I), \quad I > 0 \tag{10}$$

where the mean of RV $I$ is assumed to be unity. It can be shown that when $\alpha$ approaches $\infty$, the PDF of $K$-distributed model specialises to (10) [7, eq. (17)].

## 4 Block error rate analysis for Gamma–Gamma family turbulence models

The average BLER for SIM over a slow fading channel can be written as

$$P(M,\ N) = \int_0^\infty P(M,\ N;\ \gamma) f(\gamma)\,\mathrm{d}\gamma \tag{11}$$

where $P(M, N; \gamma)$ is the conditional block error probability and $f(\gamma)$ denotes the PDF of the instantaneous SNR. Using $\gamma = \bar{\gamma} I^2$, the PDFs of the instantaneous SNR of the Gamma–Gamma, $K$-distributed and negative exponential channels can be, respectively, shown to be (see (12))

and

$$f_K(\gamma) = \frac{1}{2\sqrt{\gamma\bar{\gamma}}} \sum_{p=0}^{\infty} \left[ a_p(\alpha,\ 1)\left(\frac{\gamma}{\bar{\gamma}}\right)^{(p/2)} + a_p(1,\ \alpha)\left(\frac{\gamma}{\bar{\gamma}}\right)^{(p+\alpha-1/2)} \right] \tag{13}$$

and

$$f_{\rm NE}(\gamma) = \frac{1}{2\sqrt{\gamma\bar{\gamma}}} \exp\left(-\sqrt{\frac{\gamma}{\bar{\gamma}}}\right) \tag{14}$$

### 4.1 NCFSK/DPSK modulation

The conditional probability of bit-error for non-coherent modulations is

$$p(\gamma) = \frac{1}{2}\exp(-\eta\gamma) \tag{15}$$

where $\eta = 1/2$ for NCFSK and $\eta = 1$ for DPSK.

For the Gamma–Gamma turbulence channel model, we substitute (1), (15), (12) into (11), and obtain the average BLER as (see (16))

Using the binomial expansion formula, we obtain

$$\left(\frac{1}{2}\mathrm{e}^{-\eta\gamma}\right)^m \left(1 - \frac{1}{2}\mathrm{e}^{-\eta\gamma}\right)^{N-m} = \sum_{k=0}^{N-m} \binom{N-m}{k} (-1)^k \left(\frac{1}{2}\right)^{m+k} \mathrm{e}^{-(m+k)\eta\gamma} \tag{17}$$

Substituting (17) into (16) and using an integral identity [17, eq. 3.326 (2.10)], we solve for the integral (see (18))

Applying (18) to (16), we obtain a series solution to the average BLER as (see (19))

The analytical result obtained in (19) is new, and it can be used to compute the BLER of SIM systems employing non-coherent modulations over the Gamma–Gamma turbulence channels. The series solution in (19) is a converging series, and the proof is provided in Appendix 1. Using (19), it is straightforward to obtain the average BLER over the $K$-distributed turbulence model by setting

$$f_{\rm GG}(\gamma) = \frac{1}{2\sqrt{\gamma\bar{\gamma}}} \sum_{p=0}^{\infty} \left[ a_p(\alpha,\ \beta)\left(\frac{\gamma}{\bar{\gamma}}\right)^{(p+\beta-1/2)} + a_p(\beta,\ \alpha)\left(\frac{\gamma}{\bar{\gamma}}\right)^{(p+\alpha-1/2)} \right] \tag{12}$$

$$\begin{aligned} P_{\rm GG}^{\rm nc}(M,\ N) = \frac{1}{2}\sum_{m=M+1}^{N}\sum_{p=0}^{\infty}\binom{N}{m}\Bigg[ & a_p(\alpha,\ \beta)\,\bar{\gamma}^{-(p+\beta/2)} \int_0^\infty \left(\frac{1}{2}\mathrm{e}^{-\eta\gamma}\right)^m \left(1 - \frac{1}{2}\mathrm{e}^{-\eta\gamma}\right)^{N-m} \\ & \times \gamma^{(p+\beta-2/2)}\,\mathrm{d}\gamma + a_p(\beta,\ \alpha)\,\bar{\gamma}^{-(p+\alpha/2)} \int_0^\infty \left(\frac{1}{2}\mathrm{e}^{-\eta\gamma}\right)^m \left(1 - \frac{1}{2}\mathrm{e}^{-\eta\gamma}\right)^{N-m} \gamma^{(p+\alpha-2/2)}\,\mathrm{d}\gamma \Bigg] \end{aligned} \tag{16}$$

$$\int_0^\infty \left(\frac{1}{2}\mathrm{e}^{-\eta\gamma}\right)^m \left(1 - \frac{1}{2}\mathrm{e}^{-\eta\gamma}\right)^{N-m} \gamma^{(p+x-2/2)}\,\mathrm{d}\gamma = \sum_{k=0}^{N-m}\binom{N-m}{k}(-1)^k\left(\frac{1}{2}\right)^{m+k} \times [\eta(m+k)]^{-(p+x/2)}\Gamma\left(\frac{p+x}{2}\right) \tag{18}$$

$$\begin{aligned} P_{\rm GG}^{\rm nc}(M,\ N) = \sum_{m=M+1}^{N}\sum_{k=0}^{N-m}\sum_{p=0}^{\infty}\binom{N}{m}\binom{N-m}{k}(-1)^k\left(\frac{1}{2}\right)^{m+k+1}\Bigg[ & a_p(\alpha,\ \beta)\,\bar{\gamma}^{-(p+\beta/2)}\,[\eta(m+k)]^{-(p+\beta/2)} \\ & \times\Gamma\left(\frac{p+\beta}{2}\right) + a_p(\beta,\ \alpha)\,\bar{\gamma}^{-(p+\alpha/2)}\,[\eta(m+k)]^{-(p+\alpha/2)}\Gamma\left(\frac{p+\alpha}{2}\right)\Bigg] \end{aligned} \tag{19}$$


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$\beta = 1$ as (see (20))

For the negative exponential turbulence channel model, substituting (1) and (14) into (11) and using the binomial expansion formula, we obtain the average BLER as (see (21))

Using [17, eq. 3.322(2)], we simplify the integral and obtain a series solution for the negative exponential channel as (see (22))

where

$$Q(x) = \int_x^\infty \exp(-u^2/2)/\sqrt{2\pi}\,\mathrm{d}u$$

is the Gaussian $Q$-function.

When $M = 0$ and $N = 1$, our BLER results in (19), (20) and (22) specialise to the BER results obtained in previous works [6, 7].

### 4.2 BPSK modulation

For coherent BPSK modulation, the conditional BER is $p(\gamma) = Q(\sqrt{2\gamma})$. Substituting this BER into (1) and using the binomial expansion formula, the average BLER can be written as (see (23))

It is challenging to find an exact expression from (23) because it involves integration of the $(m+k)$th power of the Gaussian $Q$-function. Therefore we consider an approximation of the Gaussian $Q$-function to further evaluate the average probability of error.

In the wireless communication literatures, there exist numerous approximations to the Gaussian $Q$-function. One analytically tractable approximation is the sum of exponentials approximation. The infinite sum of exponentials approximation is given by [22]

$$Q(x) = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N}\frac{1}{2}\mathrm{e}^{-\tilde{a}_i x^2} \tag{24}$$

where

$$\tilde{a}_i = (1/2)\sin^{-2}\big(\pi(i-1)/(2N-2)\big)$$

and this approximation is only accurate when $N$ is asymptotically large. Recently, based on a prony approximation approach, Loskot and Beaulieu [22] proposed two new sum of exponentials approximations involving only two terms and three terms, respectively, as

$$Q(x) \simeq 0.208\mathrm{e}^{-0.971x^2} + 0.147\mathrm{e}^{-0.525x^2} \tag{25}$$

and

$$Q(x) \simeq 0.168\mathrm{e}^{-0.876x^2} + 0.144\mathrm{e}^{-0.525x^2} + 0.002\mathrm{e}^{-0.603x^2} \tag{26}$$

It was shown in [22] that both (25) and (26) can provide adequate approximation of the Gaussian $Q$-function without having a large number of exponential terms. Using an upper bound of the complementary error function erfc($x$) and the trapezoidal rule, Chiani *et al.* [23] proposed the following two-term approximation

$$Q(x) \simeq \frac{1}{12}\exp\left(-\frac{1}{2}x^2\right) + \frac{1}{4}\exp\left(-\frac{2}{3}x^2\right) \tag{27}$$

Using an approach similar to [23], we propose the following new sum of three exponentials approximation of the $Q(x)$ as

$$Q(x) \simeq \frac{5}{24}\exp(-2x^2) + \frac{4}{24}\exp\left(-\frac{11}{20}x^2\right) + \frac{1}{24}\exp\left(-\frac{1}{2}x^2\right) \tag{28}$$

A detailed derivation of (28) is given in Appendix 2. To assess the accuracy of (25)–(28), we plot the relative error of these approximations in Fig. 1. It is shown that none of (25)–(28) can provide a uniform accurate approximation to $Q(x)$. The Prony approximations in (25) and (26) can provide better approximation for a large argument,

$$P_K^{\mathrm{nc}}(M, N) = \sum_{m=M+1}^{N}\sum_{k=0}^{N-m}\sum_{p=0}^{\infty}\binom{N}{m}\binom{N-m}{k}(-1)^k\left(\frac{1}{2}\right)^{m+k+1}\Big[a_p(\alpha, 1)\,\bar{\gamma}^{-(p+1/2)}[\eta(m+k)]^{-(p+1/2)} \times\Gamma\left(\frac{p+1}{2}\right) + a_p(1, \alpha)\,\bar{\gamma}^{-(p+\alpha/2)}[\eta(m+k)]^{-(p+\alpha/2)}\Gamma\left(\frac{p+\alpha}{2}\right)\Big] \tag{20}$$

$$P_{\mathrm{NE}}^{\mathrm{nc}}(M, N) = \sum_{m=M+1}^{N}\sum_{k=0}^{N-m}\binom{N}{m}\binom{N-m}{k}(-1)^k\left(\frac{1}{2}\right)^{m+k+1}\bar{\gamma}^{-(1/2)} \times \int_0^\infty exp\left(-\eta\gamma(m+k) - \sqrt{\frac{\gamma}{\bar{\gamma}}}\right)\gamma^{-(1/2)}\,\mathrm{d}\gamma \tag{21}$$

$$P_{\mathrm{NE}}^{\mathrm{nc}}(M, N) = \sum_{m=M+1}^{N}\sum_{k=0}^{N-m}\binom{N}{m}\binom{N-m}{k}(-1)^k\left(\frac{1}{2}\right)^{m+k}\sqrt{\frac{\pi}{\eta\bar{\gamma}(m+k)}} \times exp\left(\frac{1}{4\eta\bar{\gamma}(m+k)}\right)Q\left(\sqrt{\frac{1}{2\eta\bar{\gamma}(m+k)}}\right) \tag{22}$$

$$P^{c}(M, N) = \int_0^\infty \sum_{m=M+1}^{N}\sum_{k=0}^{N-m}\binom{N}{m}\binom{N-m}{k}(-1)^k\left(Q(\sqrt{2\gamma})\right)^{m+k} f_\gamma(\gamma)\,\mathrm{d}\gamma \tag{23}$$


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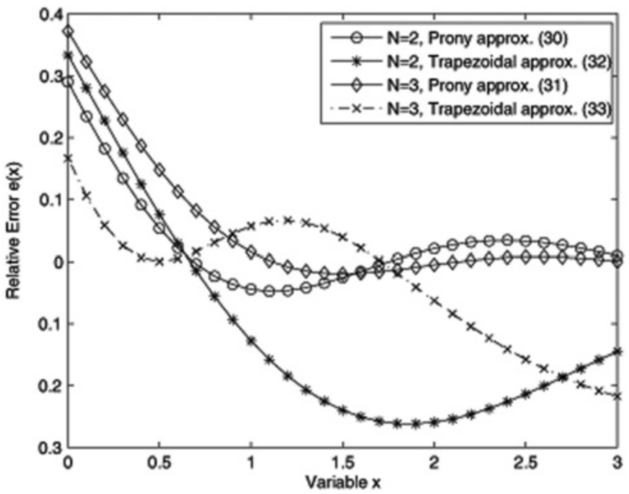


**Fig. 1** *Relative error for various sum of exponentials approximations of the Gaussian Q-function*

especially when $x > 2$, whereas our new approximation in (28) is more accurate for small argument (e.g. $x < 1$) of $Q(x)$. Since the average error rate performance is a weighted average of $Q(x)$ with respect to the fading PDF, the asymptotic large SNR error rate performance will largely depend on the behaviour of the fading channel PDF near its origin [24]. Therefore the accuracy of $Q(x)$ with smaller argument will play more significant role in the small error rate regime. For this reason, we will choose (28) to obtain a more accurate estimation of the BLER for coherent BPSK signalling.

Now substituting (28) into the $Q$-function of (23) and using the multinomial expansion formula, we have (see (29))

Finally, substituting (12) and (29) into (23) and using [17, eq. 3.381(4)], we obtain a simplified series solution to the average BLER of coherent BPSK in the Gamma–Gamma turbulence channels as (see (30))

Similarly, we can obtain the average BLER of coherent BPSK in $K$-distributed and negative exponential turbulence channels, respectively, as (see (31)) and (see (32)).

In obtaining (30)–(32), we have used an approximation for the Gaussian $Q$-function, and for $M = 0$ and $N = 1$ this approximation will result in less accurate BER estimation than the one obtained in [6, 7]. However, our BLER results are more general and can be applied to any values of $M$ and $N$.

$$\left(\frac{5}{24}\mathrm{e}^{-4\gamma}+\frac{1}{6}\mathrm{e}^{-11\gamma/10}+\frac{1}{24}\mathrm{e}^{-\gamma}\right)^{m+k}=\sum_{t=0}^{m+k}\sum_{l=0}^{t}\binom{m+k}{t}\binom{t}{l}\left(\frac{1}{24}\right)^{m+k-t}\left(\frac{5}{24}\right)^{t-l}\left(\frac{1}{6}\right)^{l}\times\exp\left(-\gamma\left(m+k+3t-\frac{29}{10}l\right)\right) \tag{29}$$

$$\begin{aligned}P_{\mathrm{GG}}^{c}(M,\,N)\simeq&\sum_{m=M+1}^{N}\sum_{k=0}^{N-m}\sum_{t=0}^{m+k}\sum_{l=0}^{t}\sum_{p=0}^{\infty}\binom{N}{m}\binom{N-m}{k}\binom{m+k}{t}\binom{t}{l}(-1)^{k}\frac{1}{2}\left(\frac{1}{24}\right)^{m+k-t}\\&\times\left(\frac{5}{24}\right)^{t-l}\left(\frac{1}{6}\right)^{l}\left[a_p(\alpha,\,\beta)\,\bar{\gamma}^{-(p+\beta/2)}\left(m+k+3t-\frac{29}{10}l\right)^{-(p+\beta/2)}\Gamma\left(\frac{p+\beta}{2}\right)\right.\\&\left.+a_p(\beta,\,\alpha)\,\bar{\gamma}^{-(p+\alpha/2)}\left(m+k+3t-\frac{29}{10}l\right)^{-(p+\alpha/2)}\Gamma\left(\frac{p+\alpha}{2}\right)\right]\end{aligned} \tag{30}$$

$$\begin{aligned}P_{K}^{c}(M,\,N)\simeq&\sum_{m=M+1}^{N}\sum_{k=0}^{N-m}\sum_{t=0}^{m+k}\sum_{l=0}^{t}\sum_{p=0}^{\infty}\binom{N}{m}\binom{N-m}{k}\binom{m+k}{t}\binom{t}{l}(-1)^{k}\frac{1}{2}\left(\frac{1}{24}\right)^{m+k-t}\\&\times\left(\frac{5}{24}\right)^{t-l}\left(\frac{1}{6}\right)^{l}\left[a_p(\alpha,\,1)\,\bar{\gamma}^{-(p+1/2)}\left(m+k+3t-\frac{29}{10}l\right)^{-(p+1/2)}\Gamma\left(\frac{p+1}{2}\right)\right.\\&\left.+\,a_p(1,\,\alpha)\,\bar{\gamma}^{-(p+\alpha/2)}\left(m+k+3t-\frac{29}{10}l\right)^{-(p+\alpha/2)}\Gamma\left(\frac{p+\alpha}{2}\right)\right]\end{aligned} \tag{31}$$

$$\begin{aligned}P_{\mathrm{NE}}^{c}(M,N)\simeq&\sum_{m=M+1}^{N}\sum_{k=0}^{N-m}\sum_{t=0}^{m+k}\sum_{l=0}^{t}\binom{N}{m}\binom{N-m}{k}\binom{m+k}{t}\binom{t}{l}(-1)^{k}\,\bar{\gamma}^{-(1/2)}\left(\frac{1}{24}\right)^{m+k-t}\\&\times\left(\frac{5}{24}\right)^{t-l}\left(\frac{1}{6}\right)^{l}\sqrt{\frac{\pi}{m+k+3t-29l/10}}\exp\left(\frac{1}{4\bar{\gamma}(m+k+3t-29l/10)}\right)\\&\times Q\left(\sqrt{\frac{1}{2\bar{\gamma}(m+k+3t-29l/10)}}\right)\end{aligned} \tag{32}$$


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### 4.3 Asymptotic error rate analysis

From the truncation error analysis (see Appendix 1), we note that our series solution is increasingly accurate in high SNR regimes. Therefore we can perform asymptotic error rate analysis to examine the behaviour of error rate in high SNR regimes. Assuming $\alpha > \beta > 0$, so the term $\bar{\gamma}^{-(p+\alpha/2)}$ decreases faster than the term $\bar{\gamma}^{-(p+\beta/2)}$ in (19) for the same $P$ value as the average SNR $\bar{\gamma}$ increases. Consequently, when $\bar{\gamma}$ approaches $\infty$, the second term in (19) can be neglected. It follows that the BLER of NCFSK/DPSK modulations in high SNR regimes can be approximated by (see (33))

Alternatively, the asymptotic BLER in (33) can be obtained by a Mellin transformation of the conditional BLER and the derivation is presented in Appendix 3.

Similarly, the asymptotic BLER of BPSK modulation can be expressed as (see (34))

From (33) and (34), we observe the diversity order of both systems is $(\beta/2)$ or $(1/2)$ min $(\alpha, \beta)$, which is the same as diversity order of the corresponding BER plots.

## 5 BLER analysis for lognormal turbulence model

The PDF of instantaneous SNR for lognormal turbulence model is given as

$$f_{\mathrm{L}}(\gamma) = \frac{1}{2\sqrt{2\pi}\sigma\gamma} \exp\left(-\frac{\left(\ln(\gamma/\bar{\gamma}) + \sigma^2\right)^2}{8\sigma^2}\right) \tag{35}$$

which is another lognormal distribution. Since the exact BLER analysis involving the lognormal turbulence PDF is analytically intractable and no closed-form exact expression of the BLER can be obtained, we propose to use the Gauss–Laguerre quadrature integration method to obtain an accurate estimation of BLER.

### 5.1 NCFSK/DPSK modulation

Substituting (17) into (11), we obtain (see (36))

To solve the integral in (36), we recall the Gauss–Laguerre quadrature integral that is given by [25, eq. 25.4.45]

$$\int_0^\infty \mathrm{e}^{-x} f(x)\, \mathrm{d}x \simeq \sum_{i=1}^{n} w_i f(x_i) \tag{37}$$

where $x_i$ is the $i$th zero of Laguerre polynomials $L_n(x)$ and $w_i = \{x_i/(n+1)^2[L_{n+1}(x_i)]^2\}$ is the associated weights. Applying (37) to (36), we can approximate the BLER of NCFSK/DPSK as (see (38))

### 5.2 BPSK modulation

To obtain the BLER of coherent BPSK over the lognormal turbulence channels, we can also use the Gauss–Laguerre quadrature integration method. Substituting (29) into (23) and using (37), we obtain the following approximate BLER of BPSK signalling as (see (39))

When $M = 0$ and $N = 1$, the BLER results in (38) and (39) specialise to the BER expressions obtained in [26].

## 6 Numerical results

In this section, we compare the BLER performance of SIM over different atmospheric turbulence models. The approximate BLERs are obtained by eliminating the infinite

$$P_{\mathrm{GG,\,asym}}^{\mathrm{nc}} = \sum_{m=M+1}^{N} \sum_{k=0}^{N-m} \binom{N}{m} \binom{N-m}{k} (-1)^k \left(\frac{1}{2}\right)^{m+k+1} a_0(\alpha, \beta)\, \bar{\gamma}^{-(\beta/2)} \left(\eta(m+k)\right)^{-(\beta/2)} \Gamma\left(\frac{\beta}{2}\right) \tag{33}$$

$$\begin{aligned} P_{\mathrm{GG,\,asym}}^{c} = {} & \sum_{m=M+1}^{N} \sum_{k=0}^{N-m} \sum_{t=0}^{m+k} \sum_{l=0}^{t} \binom{N}{m} \binom{N-m}{k} \binom{m+k}{t} \binom{t}{l} (-1)^k \frac{1}{2} \left(\frac{1}{24}\right)^{m+k-t} \\ & \times \left(\frac{5}{24}\right)^{t-l} \left(\frac{1}{6}\right)^{l} a_0(\alpha, \beta)\, \bar{\gamma}^{-(\beta/2)} \left(m + k + 3t - \frac{29}{10} l\right)^{-(\beta/2)} \Gamma\left(\frac{\beta}{2}\right) \end{aligned} \tag{34}$$

$$P_{\mathrm{L}}^{\mathrm{nc}}(M, N) = \sum_{m=M+1}^{N} \sum_{k=0}^{N-m} \binom{N}{m} \binom{N-m}{k} (-1)^k \left(\frac{1}{2}\right)^{m+k} \int_0^\infty \exp\left(-\gamma\eta(m+k)\right) f_{\mathrm{L}}(\gamma)\, \mathrm{d}\gamma \tag{36}$$

$$P_{\mathrm{L}}^{\mathrm{nc}}(M, N) \simeq \sum_{m=M+1}^{N} \sum_{k=0}^{N-m} \sum_{i=1}^{n} \binom{N}{m} \binom{N-m}{k} (-1)^k \left(\frac{1}{2}\right)^{m+k} \frac{w_i}{\sqrt{8\pi}\sigma x_i} \times \exp\left(-\frac{\left[\ln\left(x_i/\eta\bar{\gamma}(m+k)\right) + \sigma^2\right]^2}{8\sigma^2}\right) \tag{38}$$

$$\begin{aligned} P_{\mathrm{L}}^{c}(M, N) \simeq {} & \sum_{m=M+1}^{N} \sum_{k=0}^{N-m} \sum_{t=0}^{m+k} \sum_{l=0}^{t} \sum_{i=1}^{n} \binom{N}{m} \binom{N-m}{k} \binom{m+k}{t} \binom{t}{l} (-1)^k \left(\frac{1}{24}\right)^{m+k-t} \left(\frac{5}{24}\right)^{t-l} \\ & \times \left(\frac{1}{6}\right)^{l} \frac{w_i}{\sqrt{8\pi}\sigma x_i} \exp\left(-\frac{\left[\ln\left(x_i/\bar{\gamma}(m+k+3t-29l/10)\right) + \sigma^2\right]^2}{8\sigma^2}\right) \end{aligned} \tag{39}$$




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terms after the first $L+1$ terms in the series solutions. We have chosen $L=60$ for all numerical results. We consider weak ($\alpha=3.78$, $\beta=3.74$), moderate ($\alpha=2.5$, $\beta=2.06$) and strong ($\alpha=2.04$, $\beta=1.1$) turbulence conditions when describing the Gamma–Gamma turbulence models. The exact BLER curves are obtained by numerical integration using (1) and (16) along with the appropriate PDFs of SNR.

Fig. 2 shows the BLER for NCFSK over several representative block lengths. We compare the BLER in both fading and non-fading channels. Fig. 2 clearly shows the impact of turbulence fading on the BLER. The performance of BLER under a non-fading channel improves rapidly with electrical SNR, whereas fading can degrade the BLER performance significantly. We also note a larger block length tends to give higher BLER.

In Fig. 3, BLERs are illustrated for NCFSK over the Gamma–Gamma channels with various turbulence conditions. The results demonstrate excellent agreement between the exact BLERs and our series solutions. We also observe that the asymptotic BLER approaches the exact BLER faster for strong turbulence condition ($\alpha=2.04$, $\beta=1.10$). This is because the asymptotic BLERs are determined only by the smaller channel parameter $\beta$ at high SNR.

In Fig. 4, we present BLERs for NCFSK over the $K$-distributed turbulence channels with different $\alpha$ values. The results also show excellent agreement between the exact BLERs and the series solutions with $L=60$, where the exact BLER is calculated by numerical integration of (1) and (8). We observe that the slope of $K$-distributed models is $-1/2$ which is determined by $\beta=1$.

Fig. 5 demonstrates the BLER of BPSK signalling over the Gamma–Gamma channels. Again, our approximate solutions have excellent agreement with the exact BLER when we make a use of the three-term approximation of the Gaussian $Q$-function in (29). As expected, the BLER performance is better for weak turbulence condition. For example, when SNR = 30 dB, the BLER is at $7\times10^{-4}$ for $\alpha=3.78$ and $\beta=3.74$, and this BLER degrades to $5\times10^{-2}$ under a strong turbulence environment for $\alpha=2.50$ and $\beta=2.06$.

Finally, the BLER based on a Gauss–Laguerre quadrature approximation is plotted in Fig. 6 for the lognormal turbulence channel with $\sigma=0.2$. Here, we use the fifth-order Gauss–Laguerre approximation to achieve an

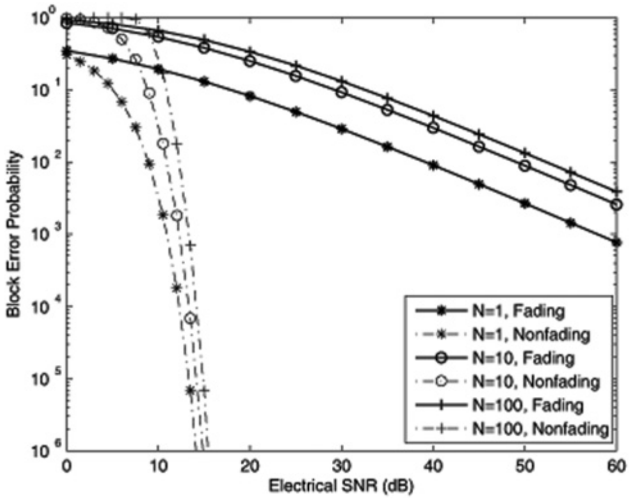


**Fig. 2** *BLER of NCFSK, $P^{nc}_{\mathrm{GG}}(0, N)$, over an unfaded and a faded Gamma–Gamma channel when $\alpha=2.04$ and $\beta=1.10$*

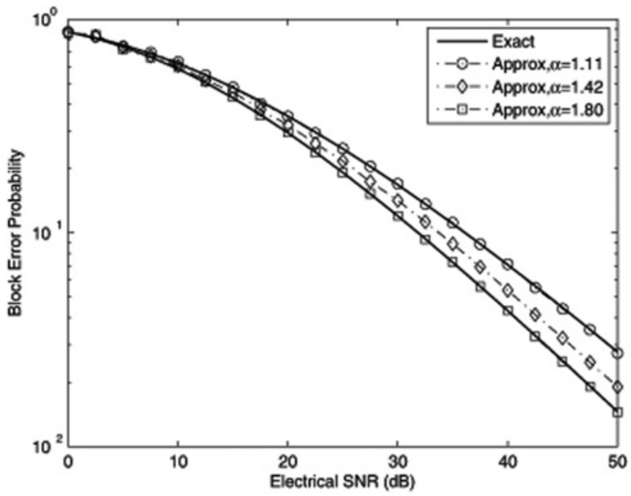


**Fig. 4** *BLER of NCFSK, $P^{nc}_{K}(2, 53)$, over the K-distributed channels with different turbulence levels*

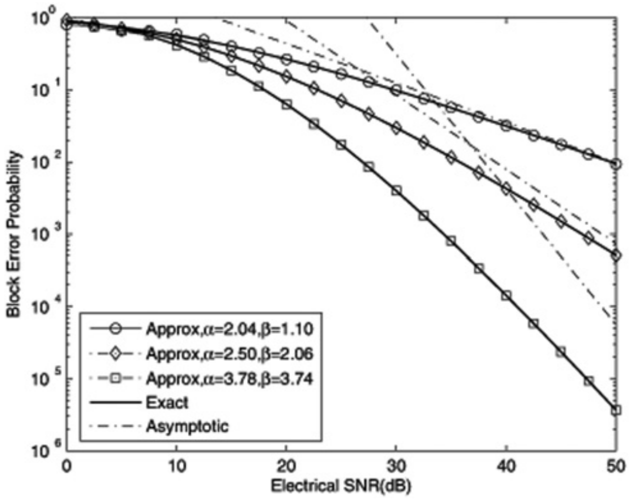


**Fig. 3** *BLER of NCFSK, $P^{nc}_{\mathrm{GG}}(2, 53)$, over the Gamma–Gamma channels with different turbulence levels*

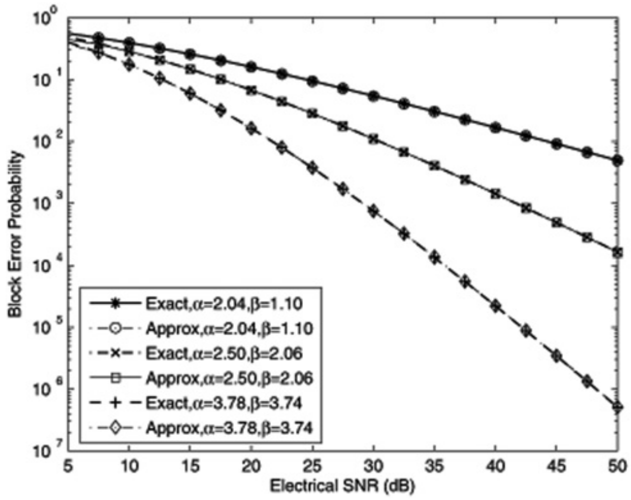


**Fig. 5** *BLER of BPSK signalling, $P^{c}_{\mathrm{GG}}(2, 53)$, over the Gamma–Gamma channels with different turbulence levels*


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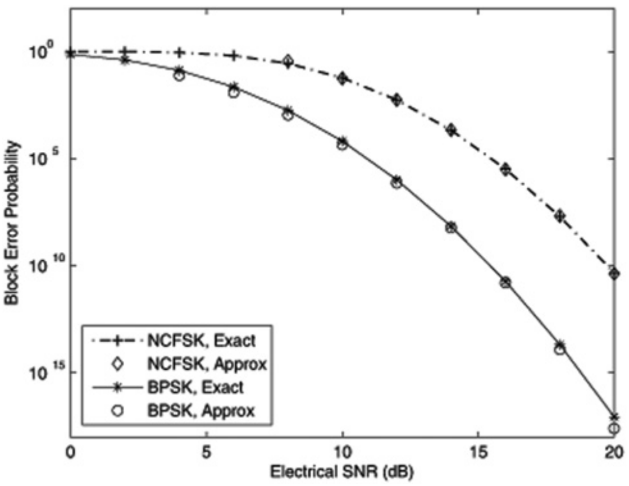


**Fig. 6** *BLER of NCFSK and BPSK, $P_L^{nc}(2, 53)$ and $P_L^{c}(2, 53)$ over the lognormal channel with $\sigma = 0.2$ using a Laguerre polynomial of order $n = 5$*

accurate estimation of the exact BLER for both NCFSK and BPSK modulations. Our approximation solutions agree well with the exact BLERs over a wide range of SNR values, where the exact result is obtained by numerical integrations of (1) and (35).

## 7 Conclusion

The BLER is a meaningful performance metric for OWC systems since atmospheric turbulence channels typically change slowly. In this paper, we presented an analytical study on BLER for SIM-based OWC systems over a variety of atmospheric turbulence channels. Our BLER expressions specialise to closed-form BER expressions for the Gamma–Gamma, the *K*-distributed and the negative exponential channels. Asymptotic BLER analysis is also conducted; we find that the asymptotic BLER approaches the exact BLER faster for stronger turbulence condition. The diversity order of BLER is the same as that of BER obtained in previous works. It has been demonstrated that the proposed analytical solutions can provide highly accurate BLER estimation and can be a useful analytical tool for OWC system designers.

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## 9 Appendix

### 9.1 Appendix 1: truncation error analysis

The truncation error caused by eliminating the infinite terms after the $L + 1$ terms in (19) can be defined as (see (40))

$$\epsilon_{\mathrm{L}} = \sum_{m=M+1}^{N} \sum_{k=0}^{N-m} \sum_{p=L+1}^{\infty} \binom{N}{m} \binom{N-m}{k} (-1)^k \left(\frac{1}{2}\right)^{m+k+1} \frac{1}{p!} \left(\frac{\alpha\beta}{\sqrt{\eta\bar{\gamma}(m+k)}}\right)^p \times \left[u_{p1}(\alpha,\ \beta) + u_{p1}(\beta,\ \alpha)\right] \tag{40}$$


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where

$$u_{p1}(x, y) = \frac{\Gamma(x-y)\Gamma(1-x+y)}{\Gamma(x)\Gamma(y)\Gamma(p-x+y+1)} \left(\frac{xy}{\sqrt{\eta\bar{\gamma}(m+k)}}\right)^y \tag{41}$$

Using the Taylor series expansion of exponential function, we can simplify the summation term in (40) to be

$$\sum_{p=L+1}^{\infty} \frac{1}{p!} \left(\frac{\alpha\beta}{\sqrt{\eta\bar{\gamma}(m+k)}}\right)^p = \exp\left(\frac{\alpha\beta}{\sqrt{\eta\bar{\gamma}(m+k)}}\right) \tag{42}$$

We then obtain an upper bound of the truncation error of BLER employing NCFSK in (19) as (see (43))

After examining the first term in (41), we find that $u_{p1}(\alpha, \beta)$ or $u_{p1}(\beta, \alpha)$ approaches zero as $p$ tends to infinite. Therefore the truncation error $\epsilon_L$ diminishes with increasing index $p$. Similarly, we can obtain an upper bound of the truncation error for BLER employing BPSK in (30) as (see (44))

where

$$u_{p2}(x, y) = \frac{\Gamma(x-y)\Gamma(1-x+y)}{\Gamma(x)\Gamma(y)\Gamma(p-x+y+1)} \times \left(\frac{xy}{\sqrt{\bar{\gamma}(m+k+3t-(29/10)l)}}\right)^y \tag{45}$$

### 9.2 Appendix 2: three-term sum of exponentials approximation of Q-function

The Gaussian $Q$-function can be expressed in terms of the complementary error function erfc(·) as

$$Q(x) = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{46}$$

In the following, we will first focus on the approximation of erfc(·). From [27], we have the following integral of an exponential form for the erfc($x$) as

$$\operatorname{erfc}(x) = \frac{2}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{\sin^2\theta}\right) d\theta, \quad x \geq 0 \tag{47}$$

It is observed that $\exp[-(x^2/\sin^2\theta)]$ is a monotonically increasing function in $\theta$ for $0 \leq \theta \leq \pi/2$. We can arbitrarily choose $N+1$ values of $\theta$ between 0 and $\pi/2$, $0 = \theta_0 \leq \theta_1 \ldots \leq \theta_N = \pi/2$, to obtain the following exponential upper bound as

$$\operatorname{erfc}(x) \leq \frac{2}{\pi}\sum_{i=1}^{N}\int_{\theta_{i-1}}^{\theta_i} \exp\left(-\frac{x^2}{sin^2\theta_i}\right) d\theta \tag{48}$$

We choose $N = 3$ and let $\boldsymbol{\theta} = [\theta_1, \theta_2]$. Making a use of the trapezoidal rule, we obtain (see (49))

Parameters $\theta_1$ and $\theta_2$ can be chosen to minimise the integral of the relative error over a specified range [0, $R$], that is

$$\boldsymbol{\theta}_{\text{opt}} = \arg\min_{\theta_1, \theta_2} \frac{1}{R}\int_0^R \frac{|g(x, \boldsymbol{\theta}) - \operatorname{erfc}(x)|}{\operatorname{erfc}(x)} dx \tag{50}$$

where the optimum values are chosen in the range from 0 to $R = 13$ dB [23]. Substituting (47) and (49) into (50), we find $(\theta_1)_{\text{opt}} \simeq \pi/6$ and$(\theta_2)_{\text{opt}} \simeq 5\pi/12$. Therefore we obtain the following approximation (see (51))

From (46), we derive a new sum of three exponentials approximation of the Gaussian $Q$-function as (see (52))

### 9.3 Appendix 3: asymptotic BLER using Mellin transform

An alternative method to obtain the BLER of NCFSK/DPSK in high SNR regimes is to use the Mellin transform of the conditional BLER. Recall the PDF of the optical irradiance

$$\epsilon_L \leq \sum_{m=M+1}^{N}\sum_{k=0}^{N-m}\binom{N}{m}\binom{N-m}{k}\left(\frac{1}{2}\right)^{m+k+1}\max_{p>L}[u_{p1}(\alpha, \beta) + u_{p1}(\beta, \alpha)]\exp\left(\frac{\alpha\beta}{\sqrt{\eta\bar{\gamma}(m+k)}}\right) \tag{43}$$

$$\begin{aligned}\epsilon_L \leq &\sum_{m=M+1}^{N}\sum_{k=0}^{N-m}\sum_{t=0}^{m+k}\sum_{l=0}^{t}\binom{N}{m}\binom{N-m}{k}\binom{m+k}{t}\binom{t}{l}\frac{1}{2}\left(\frac{1}{24}\right)^{m+k-t}\left(\frac{5}{24}\right)^{t-l}\left(\frac{1}{6}\right)^{l}\\ &\times \max_{p>L}[u_{p2}(\alpha, \beta) + u_{p2}(\beta, \alpha)]\exp\left(\frac{\alpha\beta}{\sqrt{\bar{\gamma}(m+k+3t-(29/10)l)}}\right)\end{aligned} \tag{44}$$

$$\operatorname{erfc}(x) \simeq g(x, \boldsymbol{\theta}) = \frac{\theta_2}{\pi}\exp\left(-\frac{x^2}{\sin^2\theta_1}\right) + \left(\frac{1}{2} - \frac{\theta_1}{\pi}\right)\exp\left(-\frac{x^2}{\sin^2\theta_2}\right) + \left(\frac{1}{2} - \frac{\theta_2}{\pi}\right)\exp(-x^2) \tag{49}$$

$$\operatorname{erfc}(x) \simeq g(x, \boldsymbol{\theta}_{\text{opt}}) = \frac{5}{12}\exp(-4x^2) + \frac{1}{3}\exp\left(-\frac{11}{10}x^2\right) + \frac{1}{12}\exp(-x^2) \tag{51}$$

$$Q(x) \simeq \frac{5}{24}\exp(-2x^2) + \frac{4}{24}\exp\left(-\frac{11}{20}x^2\right) + \frac{1}{24}\exp\left(-\frac{1}{2}x^2\right) \tag{52}$$


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$I$ is given by (6). Letting $Y = I^2$, the PDF of $Y$ is

$$f_Y(y) = \frac{1}{2}\sum_{p=0}^{\infty}\Big[a_p(\alpha,\ \beta)y^{(p+\beta-2/2)} + a_p(\beta,\ \alpha)y^{(p+\alpha-2/2)}\Big] \tag{53}$$

The instantaneous SNR is $\gamma = \bar{\gamma}Y$. The PDF $f_Y(y)$ near the origin (i.e. $y \to 0^+$) can be approximated by

$$f_Y(y) = ay^t + o(y^{t+\epsilon}) \tag{54}$$

where $\epsilon > 0$ and $a$ is a positive constant. The parameter $t$ represents the order of smoothness of $f_Y(y)$ at the origin, and both $a$ and $t$ can be determined by the PDF $f_Y(y)$.

Assuming $\beta < \alpha$, as $y \to 0^+$, we can neglect the second term of (53) and approximate $f_Y(y)$ as

$$f_Y(y) \simeq \frac{1}{2}a_0(\alpha,\ \beta)y^{(\beta-2/2)} \tag{55}$$

Comparing (55) with (54), we obtain $a = (1/2)a_0(\alpha,\ \beta)$ and $t = (\beta - 2/2)$.

The Mellin transform of the conditional BLER in (1) can be written as (see (56))

where $H(s)$ is the Mellin transform of $h(x)$ and $h(x)$ is the conditional probability of block error given by

$$h(x) = \sum_{m=M+1}^{N}\binom{N}{m}\left(\frac{1}{2}\mathrm{e}^{-\eta x}\right)^m\left(1 - \frac{1}{2}\mathrm{e}^{-\eta x}\right)^{N-m} \tag{57}$$

Using [17, eq. 3.381(4)], we can simplify (56) as (see (58))

According to Proposition 1 described in [28], the asymptotic BLER in high SNR regimes can be approximated by

$$P^{nc}_{\mathrm{GG,\,asym}} \simeq \frac{aH(t+1)}{\bar{\gamma}^{t+1}} = \frac{1}{2}a_0(\alpha,\ \beta)\frac{H(\beta/2)}{\bar{\gamma}^{(\beta/2)}} \tag{59}$$

Substituting (58) into (59), we can also obtain (33).

$$H(s) = \sum_{m=M+1}^{N}\sum_{k=0}^{N-m}\binom{N}{m}\binom{N-m}{k}(-1)^k\left(\frac{1}{2}\right)^{m+k}\int_0^{\infty} x^{s-1}\mathrm{e}^{-x\eta(m+k)}\,\mathrm{d}x \tag{56}$$

$$H(s) = \sum_{m=M+1}^{N}\sum_{k=0}^{N-m}\binom{N}{m}\binom{N-m}{k}(-1)^k\left(\frac{1}{2}\right)^{m+k}\left[\frac{1}{\eta(m+k)}\right]^s\Gamma(s) \tag{58}$$



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