

Solutions to Integrals Involving the Marcum Q -Function and Applications

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Abstract—Novel analytic solutions are derived for integrals that involve the generalized Marcum Q -function, exponential functions and arbitrary powers. Simple closed-form expressions are also derived for specific cases of the generic integrals. The offered expressions are both convenient and versatile, which is particularly useful in applications relating to natural sciences and engineering, including wireless communications and signal processing. To this end, they are employed in the derivation of the average probability of detection in energy detection of unknown signals over multipath fading channels as well as of the channel capacity with fixed rate and channel inversion in the case of correlated multipath fading and switched diversity.

Index Terms—Capacity, correlation, energy detection, marcum Q function, special functions, switch-and-stay combining.

I. INTRODUCTION

THE generalized Marcum Q -function, $Q_m(a, b)$, has been extensively involved in numerous areas of wireless communications including digital communications over fading channels, information-theoretic analysis of multi-antenna systems, cognitive radio communications, radar systems, [1]–[8] and references therein. Furthermore, its use has enabled the derivation of several tractable analytic expressions for important performance measures in communication theory [4].

The derivation of tractable analytic expressions in natural sciences and engineering is typically a tedious, if not impossible, task because cumbersome integrals are often encountered [9]–[22]. This is also the case when the Marcum Q -function

is involved in integrands along with exponential functions and arbitrary power terms. Two such integrals are the following:

$$\mathcal{G}(k, m, a, b, p) = \int_0^\infty x^{k-1} Q_m(a, b\sqrt{x}) e^{-px} dx \quad (1)$$

and

$$\mathcal{F}(k, m, a, b, p) = \int_0^\infty x^{k-1} Q_m(a\sqrt{x}, b) e^{-px} dx. \quad (2)$$

These integrals have been widely employed in the analysis of multi-channel receivers with non-coherent and differentially coherent detection as well as in the detection of unknown signals in cognitive radio and radar systems [9], [23]–[36] and the references therein. Based on this, a recursive formula for (2), that is restricted to only integer values of k and m , was firstly reported in [2]. Likewise, exact infinite series for (1) and (2) were proposed in [16] while a closed-form solution to (2) for integer values of k was recently reported in [21].

Nevertheless, the existing expressions for (1) and (2) are subject to validity restrictions, which limit the generality of the involved parameters and often render them inconvenient to use in applications of interest. Motivated by this, the present work is devoted to the derivation of novel closed-form expressions for (1) and (2), which are more generic and have a relatively tractable algebraic representation. These characteristics render them useful in several analyses in natural sciences and engineering, including the broad areas of wireless communications and signal processing. To this end, they are subsequently employed in the derivation of closed-form expressions for the following indicative applications: *i*) the average probability of detection in energy detection over Nakagami- m multipath fading channels which, unlike previous analyses, is valid for arbitrary values of m ; *ii*) the channel capacity with channel inversion and fixed rate in arbitrarily correlated Nakagami- m fading conditions using switch-and-stay combining. The derived expressions are validated extensively through comparisons with respective computer simulations results.

II. ANALYTIC SOLUTIONS TO INTEGRALS INVOLVING POWER, EXPONENTIAL AND MARCUM Q -FUNCTIONS

A. Closed-form Solutions to $\mathcal{G}(k, m, a, b, p)$

Theorem 1: For $a, b \in \mathbb{R}$, $m \in \mathbb{N}$ and $k, p \in \mathbb{R}^+$, the following closed-form representation is valid

$$\begin{aligned} \mathcal{G}(k, m, a, b, p) &= \frac{\Gamma(k)}{p^k} - \left(\frac{2}{b^2 + 2p} \right)^k e^{-\frac{a^2}{2}} \Gamma(k) \\ &\times \left[\Phi_1 \left(k, 1, 1; \frac{b^2}{b^2 + 2p}, \frac{a^2 b^2}{2b^2 + 4p} \right) \right. \\ &\left. - \sum_{n=0}^{m-1} \frac{(k)_n}{n!} \left(\frac{b^2}{b^2 + 2p} \right)^n {}_1F_1 \left(k+n; n+1; \frac{a^2 b^2}{2b^2 + 4p} \right) \right] \end{aligned} \quad (3)$$

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where $\Gamma(\cdot)$, $\Phi_1(\cdot)$ and ${}_1F_1(\cdot)$ denote the Euler Gamma function, the Humbert hypergeometric function of the first kind and the Kummer hypergeometric function, respectively [37].

Proof: The series in [16, eq. (10)] can be re-written as follows¹

$$\mathcal{G}(k, m, a, b, p) = \sum_{n=0}^{m-1} \frac{\Gamma(k+n) {}_1F_1\left(k+n; n+1; \frac{a^2 b^2}{2b^2+4p}\right)}{n! b^{-2n} 2^{-k} (b^2+2p)^{k+n} e^{\frac{a^2}{2}}} + \frac{\Gamma(k)}{p^k} - \underbrace{\frac{2^k}{e^{\frac{a^2}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma(k+n) b^{2n} {}_1F_1\left(k+n; n+1; \frac{a^2 b^2}{2b^2+4p}\right)}{n! (b^2+2p)^{k+n}}}_{\mathcal{A}(k, a, b, p)}. \tag{4}$$

Using the infinite series in [38, eq. (9.14.1)], it follows that

$$\mathcal{A}(k, a, b, p) = \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{a^{2l} b^{2n+2l} \Gamma(n+k) e^{-\frac{a^2}{2}} (k+n)_l}{n! l! 2^{l-k} (b^2+2p)^{n+k+l} (1+n)_l} \tag{5}$$

where $(x)_n$ denotes the Pochhammer symbol. Given that $(k+n)_l = (k)_{n+l} / (k)_n$ and $(1+n)_l = (1)_{n+l} / (1)_n$, one obtains

$$\mathcal{A}(k, a, b, p) = \mathcal{B} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(k)_{n+l} (1)_n \left(\frac{b^2}{b^2+2p}\right)^n \left(\frac{a^2 b^2}{2b^2+4p}\right)^l}{(1)_{n+l} n! l!} \tag{6}$$

where $\mathcal{B} = 2^k \Gamma(k) \exp(a^2/2) / (b^2+2p)^k$. Notably, eq. (6) can be expressed in terms of the Humbert function, Φ_1 , namely

$$\mathcal{A}(k, a, b, p) = \frac{2^k \Gamma(k) e^{-\frac{a^2}{2}}}{(b^2+2p)^k} \Phi_1\left(k, 1, 1; \frac{b^2}{b^2+2p}, \frac{a^2 b^2}{2b^2+4p}\right) \tag{7}$$

Inserting (7) in (4) yields (3), which completes the proof. \square

It is noted that Humbert functions and their properties have been studied extensively over the past decades [37], [38].

Theorem 2: For $a, b \in \mathbb{R}$ and $m, p \in \mathbb{R}^+$ and $k \in \mathbb{N}$, the following closed-form expression is valid

$$\mathcal{G}(k, m, a, b, p) = \frac{\Gamma(k)}{p^k} - \frac{\Gamma(k) b^{2m}}{p^k (b^2+2p)^m} e^{-\frac{a^2}{2}} \times \sum_{l=0}^{k-1} \frac{(m)_l (2p)^l}{l! (b^2+2p)^l} {}_1F_1\left(l+m; m; \frac{a^2 b^2}{2b^2+4p}\right). \tag{8}$$

Proof: By integrating (1) by parts, it follows that

$$\mathcal{G}(k, m, a, b, p) = \lim_{x \rightarrow \infty} g(x) \int \frac{x^k dx}{x e^{px}} - \lim_{x \rightarrow 0} g(x) \int \frac{x^k dx}{x e^{px}} - \int_0^{\infty} \left[\int x^{k-1} e^{-px} dx \right] \frac{d}{dx} Q_m(a, b\sqrt{x}) dx \tag{9}$$

where $g(x) = Q_m(a, b\sqrt{x})$. The integrals in (9) can be expressed in terms of the incomplete gamma function yielding

$$\mathcal{G}(k, m, a, b, p) = \lim_{x \rightarrow \infty} g(x) \frac{\Gamma(k, px)}{p^k} - \lim_{x \rightarrow 0} g(x) \frac{\Gamma(k, px)}{p^k} - \frac{1}{p^k} \int_0^{\infty} \Gamma(k, px) \frac{d}{dx} Q_m(a, b\sqrt{x}) dx. \tag{10}$$

¹Equation (10) in [16] contains a typo since the (a^2+2p) term in the denominator should read as (b^2+2p) . This has been corrected in (4).

Recalling the identities $\Gamma(a, \infty) = 0$, $\Gamma(a, 0) = \Gamma(a)$ and $Q_m(a, 0) = 1$ as well as setting $u = \sqrt{x}$ in (10) along with utilizing [2, eq. (9)] and [38, eq. (8.352.4)], it follows that

$$\mathcal{G}(k, m, a, b, p) = \frac{\Gamma(k)}{p^k} \times \left[1 - a^{1-m} e^{-\frac{a^2}{2}} \sum_{l=0}^{k-1} \frac{p^l}{l! b^{2l}} \times \int_0^{\infty} x^{m-2l} e^{-\left(\frac{p}{b^2} + \frac{1}{2}\right)x^2} I_{m-1}(ax) dx \right]. \tag{11}$$

The above integral can be expressed in closed-form using [38, eq. (6.621.1)]. To this effect, eq. (11) is deduced, which completes the proof. \square

B. Closed-form Solutions to $\mathcal{F}(k, m, a, b, p)$

A closed-form expression for (2) was reported in [21] for the case that k is integer and m is arbitrary real, namely

$$\mathcal{F}(k, m, a, b, p) = \frac{\Gamma(k) \Gamma\left(m, \frac{b^2}{2}\right)}{p^k \Gamma(m)} + \frac{a^2 b^{2m} \Gamma(k) e^{-\frac{b^2}{2}}}{m! p^k 2^m (a^2+2p)} \times \sum_{l=0}^{k-1} \left(\frac{2p}{a^2+2p}\right)^l {}_1F_1\left(l+1; m+1; \frac{a^2 b^2}{2a^2+4p}\right). \tag{12}$$

Likewise, in what follows we derive a closed-form expression for the useful case that m is integer and k is arbitrary real.

Theorem 3: For $k, p \in \mathbb{R}^+$, $m \in \mathbb{N}$ and $a, b \in \mathbb{R}$, the following closed-form expression is valid

$$\mathcal{F}(k, m, a, b, p) = \frac{\Gamma(k)}{p^k} - \frac{2^k \Gamma(k) e^{-\frac{b^2}{2}} \Phi_2\left(1, k, 1; \frac{b^2}{2}, \frac{a^2 b^2}{2a^2+4p}\right)}{(a^2+2p)^k} + \frac{\Gamma(k) 2^k e^{-\frac{b^2}{2}}}{(a^2+2p)^k} \sum_{n=0}^{m-1} \frac{b^{2n}}{n! 2^n} {}_1F_1\left(k; n+1; \frac{a^2 b^2}{2a^2+4p}\right) \tag{13}$$

where $\Phi_2(\cdot)$ is the Humbert function of the second kind [38].

Proof: The series in [16, eq. (9)] can be also expressed as

$$\mathcal{F}(k, m, a, b, p) = \frac{\Gamma(k)}{p^k} + \sum_{n=0}^{m-1} \frac{\Gamma(k) {}_1F_1\left(k; n+1; \frac{a^2 b^2}{2a^2+4p}\right)}{n! b^{-2n} 2^{-k} (a^2+2p)^k e^{\frac{b^2}{2}}} - \underbrace{\frac{\Gamma(k) 2^k e^{-\frac{b^2}{2}}}{(a^2+2p)^k} \sum_{n=0}^{\infty} \frac{b^{2n}}{n! 2^n} {}_1F_1\left(k; n+1; \frac{a^2 b^2}{2a^2+4p}\right)}_{\mathcal{H}(k, a, b, p)}. \tag{14}$$

Applying [38, eq. (9.14.1)] in $\mathcal{H}(k, a, b, p)$ and given that $(n+1)_l = \Gamma(n+l+1) / \Gamma(n+1) = (1)_{n+l} / (1)_n$, one obtains

$$\mathcal{H}(k, a, b, p) = \frac{e^{-\frac{b^2}{2}} 2^k \Gamma(k)}{(a^2+2p)^k} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \frac{(1)_n (k)_l}{(1)_{n+l}} \frac{b^{2n}}{n!} \frac{\left(\frac{a^2 b^2}{2a^2+4p}\right)^l}{l!}. \tag{15}$$

Notably, the above expression can be expressed in terms of the Humbert hypergeometric function of the second kind yielding

$$\mathcal{H}(k, a, b, p) = \frac{e^{-\frac{b^2}{2}} 2^k \Gamma(k)}{(a^2+2p)^k} \Phi_2\left(1, k, 1; \frac{b^2}{2}, \frac{a^2 b^2}{2a^2+4p}\right). \tag{16}$$

Inserting (16) into (14) yields (13), concluding the proof. \square

C. Specific Cases of $\mathcal{F}(k, m, a, b, p)$ and $\mathcal{G}(k, m, a, b, p)$

Simple expressions are derived for $k = 1$, $a = 0$ and $b = 0$.

1) *The Case That $k = 1$* : In this special case it follows that

$$\mathcal{G}(1, m, a, b, p) = \int_0^\infty Q_m(a, b\sqrt{x})e^{-px} dx \quad (17)$$

and

$$\mathcal{F}(1, m, a, b, p) = \int_0^\infty Q_m(a\sqrt{x}, b)e^{-px} dx. \quad (18)$$

Equation (17) is given by [16, eq. (16)]. In the same context, a generic closed-form expression for (18) is derived below.

Lemma 1: For $a, m \in \mathbb{R}$ and $b, p \in \mathbb{R}^+$, the following closed-form expression is valid²

$$\mathcal{F}(1, m, a, b, p) = \frac{\Gamma\left(m, \frac{b^2}{2}\right)}{p\Gamma(m)} + \frac{a^2 e^{-\frac{pb^2}{a^2+2p}} \gamma\left(m, \frac{a^2 b^2}{2a^2+4p}\right)}{pa^{2m}\Gamma(m)(a^2+2p)^{1-m}} \quad (19)$$

where $\gamma(a, x)$ denotes the lower incomplete gamma function.

Proof: By setting $k = 1$ in (12), it immediately follows that

$$\mathcal{F}(1, m, a, b, p) = \frac{\Gamma\left(m, \frac{b^2}{2}\right)}{p\Gamma(m)} + \frac{a^2 b^{2m} {}_1F_1\left(1; m+1; \frac{a^2 b^2}{2a^2+4p}\right)}{m! p 2^m (a^2+2p) e^{\frac{b^2}{2}}} \quad (20)$$

Notably, using the following hypergeometric function identity

$${}_1F_1(1; n; x) = x^{1-n} (n-1) e^x \gamma(n-1, x) \quad (21)$$

and recalling that $\gamma(a, x) = \Gamma(a) - \Gamma(a, x)$ and $m!/m = (m-1)! = \Gamma(m)$, eq. (20) can be equivalently expressed as

$$\begin{aligned} \mathcal{F}(1, m, a, b, p) &= \frac{\Gamma\left(m, \frac{b^2}{2}\right)}{p\Gamma(m)} + \frac{(a^2+2p)^{m-1} e^{-\frac{a^2 b^2}{a^2+2p}}}{pa^{2m-2}} \\ &\quad - \frac{(a^2+2p)^{m-1} e^{-\frac{a^2 b^2}{a^2+2p}}}{pa^{2m-2}\Gamma(m)} \Gamma\left(m, \frac{a^2 b^2}{2a^2+4p}\right). \end{aligned} \quad (22)$$

To this effect and performing long but basic algebraic manipulations (22) reduces to (19), which completes the proof. \square

Remark 1: A similar expression can be obtained through (13).

2) *The Case That $a = 0$* : In this special case it follows that

$$\mathcal{G}(k, m, 0, b, p) = \int_0^\infty x^{k-1} Q_m(0, b\sqrt{x})e^{-px} dx \quad (23)$$

and

$$\mathcal{F}(k, m, 0, b, p) = Q_m(0, b) \int_0^\infty x^{k-1} e^{-px} dx. \quad (24)$$

Lemma 2: For $k, p \in \mathbb{R}^+$, $m \in \mathbb{N}$ and $b \in \mathbb{R}$, the following closed-form representations are valid

$$\mathcal{G}(k, m, 0, b, p) = \frac{2^k}{(b^2+2p)^k} \sum_{l=0}^{m-1} \frac{b^{2l} \Gamma(k+l)}{l!(b^2+2p)^l} \quad (25)$$

and

$$\mathcal{F}(k, m, 0, b, p) = \frac{\Gamma(k)}{p^k} e^{-\frac{b^2}{2}} \sum_{l=0}^{m-1} \frac{b^{2l}}{l! 2^l}. \quad (26)$$

²The integral in [20, eq. (1)] was recently evaluated in closed-form. However, this solution does not account for $\mathcal{F}(k, m, a, b, p)$ since the two integrals would be equal only when $I_0(0) = 1$. Yet, this can be achieved for $\mu_2 = 1$ and $c = 0$, which eliminates the power term and as a consequence, the integral reduces to (18), which is simply expressed in closed-form in (19).

Proof: Applying [2, eq. (2)] in (23) one obtains

$$\mathcal{G}(k, m, 0, b, p) = \sum_{l=0}^{m-1} \frac{b^{2l}}{l! 2^l} \int_0^\infty x^{k+l-1} e^{-x\left(p+\frac{b^2}{2}\right)} dx. \quad (27)$$

Evidently, the above integral can be expressed in closed-form in terms of the Euler gamma function. This is also the case for $\mathcal{F}(k, m, 0, b, p)$ since $Q_m(0, b)$ is not a part of the integrand. As a result, by performing a necessary change of variables in [38, eq. (8.310.1)] and substituting (27) and (23) one obtains (25) and (24), respectively, which completes the proof. \square

3) *The Case That $b = 0$* : In this special case it follows that

$$\mathcal{G}(k, m, a, 0, p) = Q_m(a, 0) \int_0^\infty x^{k-1} e^{-px} dx \quad (28)$$

and

$$\mathcal{F}(k, m, a, 0, p) = \int_0^\infty x^{k-1} Q_m(a\sqrt{x}, 0) e^{-px} dx. \quad (29)$$

Lemma 3: For $p \in \mathbb{R}^+$ and $a, m, k \in \mathbb{R}$, the following simple closed-form expression is valid

$$\mathcal{G}(k, m, a, 0, p) = \mathcal{F}(k, m, a, 0, p) = \frac{\Gamma(k)}{p^k}. \quad (30)$$

Proof: The proof follows with the aid of the identity $Q_m(a, 0) \triangleq 1$, in [2, eq. (1)] as well as [38, eq. (8.310.1)]. \square

III. APPLICATIONS IN WIRELESS COMMUNICATIONS

As already mentioned, the derived expressions for (1) and (2) can be used in applications relating to natural sciences and engineering, including wireless communications and signal processing. Based on this, they are employed in the derivation of simple expressions for the detection of unknown signals in cognitive radio and radar systems as well as for the channel capacity using switched diversity. To this end, a closed-form expression is firstly derived for the average probability of detection over Nakagami- m fading channels. Likewise, a closed-form expression is derived for the channel capacity with channel inversion and fixed rate in switch-and-stay combining (SSC) under correlated Nakagami- m fading conditions.

A. Energy Detection Over Nakagami- m Fading Channels with Arbitrary Values of m

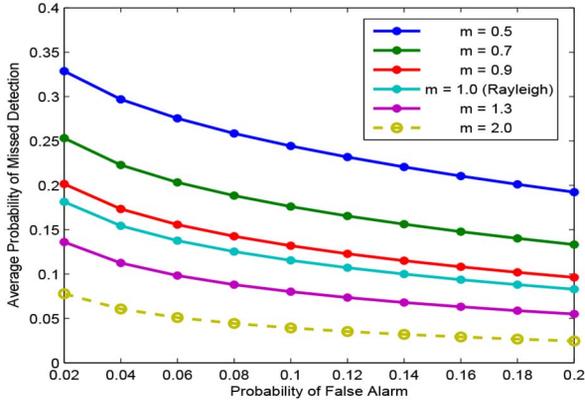
The detection of unknown signals is modeled as a binary hypothesis-testing problem, where H_0 and H_1 denote the cases that a signal is absent or present, respectively. The corresponding test statistic is typically represented by the central chi-square and the non-central chi-square distributions, respectively, and is compared with an energy threshold, λ [9].

Corollary 1: For $\bar{\gamma}, \lambda \in \mathbb{R}^+$, and either $m \geq 0.5$ and $u \in \mathbb{N}$, or $m \in \mathbb{N}$ and $u \in \mathbb{R}^+$, the average probability of detection over Nakagami- m fading channels can be expressed as

$$\bar{P}_d = \frac{m^m}{\bar{\gamma}^m \Gamma(m)} \mathcal{F}\left(m, u, \sqrt{2}, \sqrt{\lambda}, \frac{m}{\bar{\gamma}}\right). \quad (31)$$

Proof: The probability of false alarm and probability of detection in additive white Gaussian noise are given by $P_f = \Gamma(u, \lambda/2)$ and $P_d = Q_u(\sqrt{2\bar{\gamma}}, \sqrt{\lambda})$, respectively, where u and $\bar{\gamma}$ denote the time-bandwidth product and the instantaneous signal-to-noise ratio (SNR), respectively [20], [21], [23]. It is recalled that in energy detection over fading channels, the P_d is averaged over the fading statistics. To this effect, for the case Nakagami- m fading channels in [4, eq. (2.21)] is represented as follows

$$\bar{P}_d = \frac{m^m}{\bar{\gamma}^m \Gamma(m)} \int_0^\infty \gamma^{m-1} Q_u\left(\sqrt{2\bar{\gamma}}, \sqrt{\lambda}\right) e^{-\frac{m\gamma}{\bar{\gamma}}} d\gamma. \quad (32)$$


 Fig. 1. ROC curve for $u = 5$, $\bar{\gamma} = 15$ dB and different values of m .

Evidently, the above integral can be expressed in terms of (12) or (13). This yields (31), which completes the proof. \square

Notably, the offered expression can account for arbitrary values of m , contrary to existing analyses that assume integer values of m for simplicity. Fig. 1 illustrates the corresponding probability of missed detection versus probability of false alarm ROC curve for different values of m . One can notice the sensitivity of m , particularly for small values, and thus, the usefulness of the offered expression also in practical scenarios.

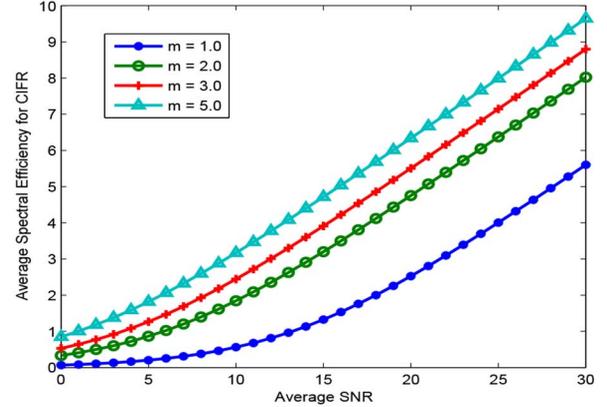
B. Capacity with Channel Inversion and Fixed Rate Over Correlated Nakagami- m Fading Using Switch-and-stay Combining

Channel capacity under different transmission policies is particularly useful in achieving certain quality of service requirements. In this context, channel inversion with fixed rate (CIFR) has been rather useful as it ensures a constant SNR at the receiver through adaptation of the transmit power. This method relies on fixed-rate modulation and fixed code design, which renders its implementation relatively simple [4].

Corollary 2: For $\bar{\gamma}, B \in \mathbb{R}^+$, $m \geq \frac{1}{2}$ and $0 \leq \rho < 1$, the capacity with channel inversion and fixed rate over correlated Nakagami- m fading channels with SSC can be expressed as

$$C_{\text{CIFR}} = B \log_2 \left(1 + \frac{1}{\mathcal{R}(m, \bar{\gamma}, \gamma_T, \rho)} \right) \quad (33)$$

where (see (34) at the bottom of the page), with ρ and γ_T denoting the correlation coefficient and the predetermined SNR switching threshold, respectively.


 Fig. 2. Average SE vs $\bar{\gamma}$ for CIFR under correlated Nakagami- m fading with SSC for $\gamma_T = 0$ dB, $\rho = 0.5$ and different values of m .

Proof: The CIFR over fading channels is defined as [39]

$$C_{\text{CIFR}} = B \log_2 \left(1 + \frac{1}{\int_0^\infty \frac{p_\gamma(\gamma)}{\gamma} d\gamma} \right). \quad (35)$$

In the case of switched diversity and correlated Nakagami- m fading, the PDF of the SSC output is given by [4, eq. (9.334)]. By also setting $\mathcal{R} = \int_0^\infty p_{\gamma_{\text{SSC}}}(\gamma)/\gamma d\gamma$ it follows that $\mathcal{R} = \int_0^\infty A(\gamma) d\gamma + \int_{\gamma_T}^\infty p_\gamma(\gamma) d\gamma$, where $A(\gamma)$ is given in [4, eq. (9.335)]. To this effect and using [4, eq. (2. 21)] one obtains (36), shown at the bottom of the page. The first two integrals in (36) can be expressed in terms of the gamma functions, whereas the third integral has the algebraic form of (2). Therefore, by performing the necessary change of variables yields (33), which completes the proof. \square

The behavior of the corresponding average spectral efficiency versus average SNR is illustrated in Fig. 2 for different values of m with fixed values of γ_T and ρ . The significant effect of the severity of fading on C_{CIFR} is clearly observed.

IV. CONCLUSION

Novel closed-form expressions were derived for two Marcum Q -function integrals which are both simple and generic. Simple analytic expressions for involved special cases were also derived in closed-form. These expressions are tractable and are expected to be useful in analyses relating to natural sciences and engineering, including wireless communications and signal processing. To this end, they were employed in the analysis of energy detection in RADAR and cognitive radio systems as well as in the channel capacity with channel inversion and fixed rate over correlated multipath fading channels.

$$\mathcal{R}(m, \bar{\gamma}, \gamma_T, \rho) = \frac{m}{\bar{\gamma}\Gamma(m)} \Gamma\left(m-1, \frac{m\gamma_T}{\bar{\gamma}}\right) - \frac{m^m}{\bar{\gamma}^m \Gamma(m)} \mathcal{F}\left(m-1, m, \sqrt{\frac{2m\rho}{(1-\rho)\bar{\gamma}}}, \sqrt{\frac{2m\gamma_T}{(1-\rho)\bar{\gamma}}}, \frac{m}{\bar{\gamma}}\right) \quad (34)$$

$$\mathcal{R}(m, \bar{\gamma}, \gamma_T, \rho) = \frac{m^m}{\bar{\gamma}^m \Gamma(m)} \int_{\gamma_T}^\infty \gamma^{m-2} e^{-\frac{m\gamma}{\bar{\gamma}}} d\gamma - m^m \int_0^\infty \frac{\gamma^{m-2} e^{-\frac{m\gamma}{\bar{\gamma}}}}{\bar{\gamma}^m \Gamma(m)} Q_m\left(\sqrt{\frac{2m\rho\gamma}{(1-\rho)\bar{\gamma}}}, \sqrt{\frac{2m\gamma_T}{(1-\rho)\bar{\gamma}}}\right) d\gamma. \quad (36)$$

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