On the Error Analysis of Hexagonal-QAM Constellations

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***Abstract*— Future communication systems are envisioned to support applications that require very high data rates in an energy efficient manner. To this direction, the use of hexagonal quadrature amplitude modulation (HQAM) provides high data rates and power efficiency, due to its compact allocation of symbols on the 2D plane. However, because of its hexagonal lattice, it is difficult, if not impossible, to evaluate exactly the error probability. In this letter, we propose a tight upper bound as well as a closed-form approximation for the symbol error probability of HQAM, which are validated from numerical and simulation results. Finally, exploiting the analytical results a novel low complexity detection scheme is presented.**

***Index Terms*— HQAM, AWGN, symbol error probability, detection scheme.**

I. Introduction

VERY high data rates and energy efficiency are of paramount importance in beyond 5G and 6G wireless communication systems. Driven by the emergence of promising applications such as virtual and augmented reality, industrial internet of things, digital twins etc., the capabilities of digital communication systems need to be expanded. This fact has led the research interest over the last few years on the investigation of higher order two-dimensional (2D) constellations, which will be able to provide high-data rate communications in an energy efficient manner.

To this direction, hexagonal quadrature amplitude modulation (HQAM) is claimed to be a modulation able to play a pivotal role in future wireless systems, by highly improving the performance in terms of data rate and energy efficiency, thanks to its compact allocation of symbols on the 2D plane [1], [2]. More precisely, HQAM, also known as triangular QAM (TQAM), has been proposed in [3], because of its densest 2D packing, which offers reduced peak and average constellation power. It should be mentioned that the symbols of an HQAM constellation are placed at the center of an equilateral hexagon. By taking into account the arrangement of the constellation symbols in relation to the origin, HQAM can be further categorized into regular HQAM (R-HQAM) and irregular HQAM (I-HQAM). Specifically, R-HQAM is symmetric around the origin [4], while in I-HQAM, the symmetry constraint is relaxed and the constellations are in circular shapes as the order of the modulation M increases [4]. As a result, I-HQAM constellations are more compact than R-HQAM. It should also be highlighted that there exists a subcategory of I-HQAM, the optimal HQAM, where both irregular structures are preserved and constellation energy is minimized [1]. Recently, a comparative study of various QAM constellations was presented justifying the supremacy of HQAM [1]. Particularly, HQAM has been investigated as a promising modulation technique, since it can provide a signal-to-noise ratio (SNR) gain with respect to square QAM (SQAM) in the high SNR regime [3]. Finally, a detection method for 16-TQAM was presented in [4].

One of the most critical metrics to characterize the performance of a modulation scheme is the symbol error probability (SEP). However, due to HQAM hexagonal lattice, it is difficult to evaluate the exact SEP. A general method to calculate the exact transition probability and bit error rate for arbitrary two-dimensional signaling formats was proposed in [5], however, its numerical integration approach led to complicated mathematical expressions. As a result, other studies on this topic provided approximation methods for SEP value. In [6] and [7], approximations for the SEP of TQAM were extracted, which are accurate only in the high SNR regime. However, both previous SEP approximations were not as tight as the method presented in [8]. Specifically, in [8], an approximation for the SEP of HQAM was provided, where the average number of nearest-neighbors was utilized and a correction term was introduced that improves the method presented in [6]. Despite its accuracy, it can be considered complex and also results for higher-order constellations were not provided. Finally, in [9], an improved approximation compared with [8] in terms of complexity was proposed, however the approximation's accuracy was decreased.

In this letter, we present an accurate and simple closed-form approximation for the SEP of HQAM with decreased complexity as well as two simple and tight upper bounds. Utilizing this approximation, a novel detection scheme of low average complexity $\mathcal{O}(\log\sqrt{M})$ is proposed. To the best of the author's knowledge, this is the first time where a detection scheme for HQAM constellations is presented that has average complexity less than $\mathcal{O}(\sqrt{M})$ [4]. Finally, we perform simulations that validate the tightness of the proposed bound and the accuracy of the approximation method and detection scheme, compared with the maximum likelihood detection (MLD).

II. System Model

An M-ary HQAM constellation, as illustrated in Fig. 1, is defined by a set $S_M = \{\mathbf{s_i} \in \mathbb{R}^2, i = 0, \ldots, M-1\}$

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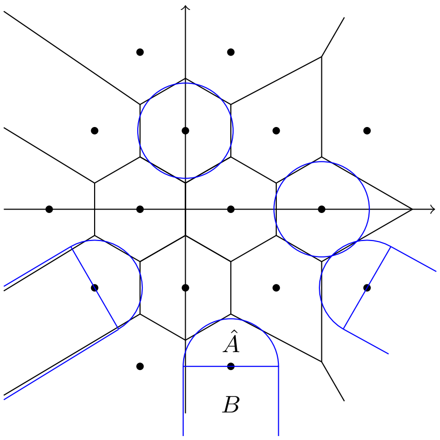

Fig. 1. 16-HQAM constellation.

of M symbols, where S_M is a subset of infinite grid $S = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \mathbf{x_0}\}$, $\mathbf{v_1} = [(d_{\min}/2), 0]^T$ and $\mathbf{v_2} = [0, (\sqrt{3}d_{\min}/2)]^T$ are the basis vectors of the 2D grid, $d_{\min}$ is the minimum distance between symbols, $c_1, c_2 \in \mathbb{Z}$ and $\mathbf{x_0} \in \mathbb{R}^2$ is the offset of the grid. With reference to the maximum-likelihood decision regions, the internal symbols are equilateral hexagons, with angles $(2\pi/3)$, while the infinite region of the external symbols have one or more angles equal to $(2\pi/3)$.

Furthermore, the received signal can be expressed as

$$r = s + n, \tag{1}$$

where s is the transmitted symbol. Moreover, the random variable (RV) n denotes the additive white Gaussian noise (AWGN) which follows the normal distribution with mean value 0 and variance $\frac{N_0}{2}$, where N_0 is the spectral density of the noise, i.e., $n \sim \mathcal{N}(0, \frac{N_0}{2})$.

III. Symbol Error Probability of HQAM

In this section, we investigate the SEP for an HQAM constellation with nearest-neighbor distance equal to $d_{\min}$ which is given by

$$d_{\min} = \sqrt{\frac{4E_s}{\alpha}}, \tag{2}$$

where E_s is the average symbol energy and α is given by [1, Table 6] for different constellation orders. Specifically, we provide an accurate SEP approximation as well as a tight upper bound by considering the incircle and the circumcircle of a regular hexagon with radii $R = \frac{d_{\min}}{2}$ and $R' = \frac{d_{\min}}{\sqrt{3}}$, respectively.

The key idea for approximating the SEP is based on the approximation of a regular hexagon's area with a circle. Specifically, it is known that the incircle and the circumcircle of a regular hexagon can be used as lower and upper bound, respectively, of its area. Therefore, an approximation for the SEP can be obtained by calculating a circle with radius ρ lying between the incircle with radius R and the circumcircle with radius R' of the hexagonal decision region. Additionally, due to the fact that the decision region of external symbols is infinite, it can be approximated by an infinite striped-shaped area which is constructed from a semicircle with radius ρ and an infinite rectangle tangential to the semicircle, as illustrated in Fig. 1.

TABLE I
Number of External Symbols b for M-HQAM

Order M	irregular	regular
16	9	10
32	13	13
64	20	22
128	29	27
256	44	46
512	62	64
1024	84	128

TABLE II
Parameter k for M-HQAM

Constellation	Order M	k
regular	16	0.8711505
regular	32	0.7233274
regular	64	0.5222431
regular	128	0.5088351
regular	256	0.3936315
regular	512	0.3672311
regular	1024	0.2982858

Theorem 1: The SEP of an HQAM constellation can be approximated as

$$P_s(\rho) = \frac{2M-b}{2M} e^{-\frac{\rho^2}{N_0}} + \frac{b}{M} Q\left(\frac{\rho\sqrt{2}}{\sqrt{N_0}}\right), \tag{3}$$

where b denotes the number of the external symbols presented in Table I,

$$\rho = kR' + (1-k)\,R \tag{4}$$

is the radius of the approximation circle with $k \in [0,1]$, and

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{u^2}{2}\right) du \tag{5}$$

is the Gaussian Q-function [10]. It should be highlighted that $k = 0$ corresponds to the incircle radius of the hexagonal decision region, while $k = 1$ corresponds to the circumcircle radius.

Proof: The proof is provided in Appendix A. ■

Remark 1: Utilizing (2) and (4), (3) can be reformulated in terms of $w = \frac{E_s}{N_0}$ and k as

$$\begin{aligned} P_s(w,k) = &\frac{2M-b}{2M} e^{-w(l_1+l_2+l_3)} \\ &+ \frac{b}{M} Q\left(\sqrt{2wl_1} + \sqrt{2wl_3}\right), \end{aligned} \tag{6}$$

where $l_1 = \frac{4k^2}{3\alpha}$, $l_2 = \frac{4k(1-k)}{\sqrt{3}\alpha}$, $l_3 = \frac{(1-k)^2}{\alpha}$.

The values of k for the SEP approximation of M-HQAM are given in Table II. Specifically, k can be obtained by solving the minimum mean square error optimization problem, i.e.,

$$k = \arg\min_k \left\{ \mathbb{E}\left[\sum_{i=1}^{N} \Big(P_s(w_i,k) - \mathrm{SEP}(w_i) \Big)^2 \right] \right\}, \tag{7}$$

where $\mathbb{E}[\cdot]$ denotes expectation, N denotes the number of selected curve samples and $\mathrm{SEP}(w_i)$ is the exact SEP for the

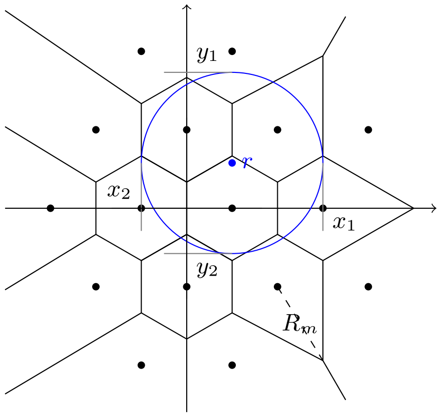

Fig. 2. Detection scheme.

i-th sample. Also, from (6), the tradeoff among the SEP, the transmitted power and the transmission rate which is expressed through the order M is observed.

Remark 2: If $k = 0$, a tight upper bound for the SEP can be derived.

Remark 3: For higher order constellations where $M \gg b$, a simpler approximation for the constellation's SEP can be expressed as

$$P_s(\rho) = e^{-\frac{\rho^2}{N_0}}. \tag{8}$$

Also, for $k = 0$, a simpler upper bound for the SEP of HQAM is given by

$$\mathrm{SEP} \leq e^{-\frac{R^2}{N_0}}. \tag{9}$$

IV. A Low Complexity Detection Scheme

In this section, we present a novel detection scheme with low complexity and performance equivalent to the MLD. The proposed detection scheme of a received symbol r is illustrated in Fig. 2 and the algorithmic procedure is presented in Algorithm 1.

Let the i-th symbol of the constellation be of the form $s_i = x_i + jy_i$, where j is the imaginary unit. All possible real values x_i of the constellation are stored in set S_x in ascending order. Moreover, A_i is the adjacent 2D-matrix of x_i given by

$$A_i = \begin{bmatrix} s_{i,1} & \mathrm{Im(s_{i,1})} \\ \vdots & \vdots \\ s_{i,p} & \mathrm{Im(s_{i,p})} \end{bmatrix}, \tag{10}$$

where $s_{i,j}$ is the j-th symbol of the constellation that satisfies $\mathrm{Re}(s_{i,j}) = x_i$, for $j = 1, \ldots, p$ and $p \leq \sqrt{M}$. A_i is sorted in ascending order based on the second column.

When a symbol $r = x_\mathrm{r} + jy_\mathrm{r}$ is received, a detection circle of radius $R_m = d_{\min}$, illustrated in Fig. 2, and center r is etched. Regions $R_1 = [x_1, x_2]$ and $R_2 = [y_1, y_2]$ are defined by the parallel to the axes tangent lines of the circle, as shown in Fig. 2 and binary search (BS) [11] is applied on the set S_x in order to find real values x_i confined in region R_1. Afterwards, for every x_i in region R_1, BS is applied to the second column of its adjacent matrix A_i, in order to find symbols $s_{i,c}$, where $\mathrm{Im}(s_{i,c}) \in R_2$. Finally, the candidate symbols $s_{i,c}$ lie inside region $G = R_1 \cap R_2$ and the decision is made by obtaining the symbol with the minimum euclidean distance from r. Finally, due to the fact that set S_x has length $2\sqrt{M}$ and the second column of A_i has length less than or equal to $\sqrt{M}$, the time complexity of BS on both of these sets is $\mathcal{O}(\log \sqrt{M})$.

Algorithm 1 Detection Algorithm

Input: Coordinates of the received symbol r
Output: Detected symbol
Etch a circle of radius R_m and center r.
Bring the tangent lines of circle R_m parallel to x axis and take the points of intersection with y axis y_1, y_2.
Bring the tangent lines of the circle R_m parallel to y axis and take the points of intersection with x axis x_1, x_2.
$\mathrm{nearestSymbol} \leftarrow \mathrm{null}$
$\mathrm{list} \leftarrow \mathrm{null}$
arrays $\mathrm{Q_i}$ are initialized
$\mathrm{x = BS(S_x, x_1, x_2)}$
for $\mathrm{x_i}$ in x **do**
  $\mathrm{s_{i,c} = BS(A_i(:,2), y_1, y_2)}$
  APPEND(list, $\mathrm{s_{i,c}}$)
end for
if list is not null **then**
  for $s_{i,c}$ in list **do**
    $\mathrm{nearestSymbol}$ = MLD($s_{i,c}$, r)
  end for
end if
if list is null **then**
  $\mathrm{i} \leftarrow$ quadrant of r
  for z in Q_i **do**
    $\mathrm{nearestSymbol}$ = MLD(z, r)
  end for
end if
return $\mathrm{nearestSymbol}$

In the low SNR regime, when region G does not enclose any of the constellation symbols, the aforementioned algorithm will not operate properly. Therefore, in this case, arrays Q_i, $i = \{1, 2, 3, 4\}$, are initialized containing the constellation's external symbols for which at least one part of their infinite decision region is located within the i-th quadrant of the 2D plane. Then, if the received symbol r lies in the i-th quadrant the decision is made by obtaining the symbol z of array Q_i with the minimum euclidean distance from r. It should be mentioned that the time complexity of linearly traversing array Q_i to find symbol z is $\mathcal{O}(\sqrt{M})$.

The average time complexity of the aforementioned algorithm is $\mathcal{O}(\log \sqrt{M})$ in high order constellations ($M \geq 512$), where $M \gg b$. Additionally, in low order constellations, where M is comparable to b, the average time complexity is $\mathcal{O}(\sqrt{M})$ in the low SNR regime due to the high possibility of G to not contain any constellation symbols. However, as SNR increases, the average time complexity tends to $\mathcal{O}(\log \sqrt{M})$, as well.

V. Simulation Results

In this section, we verify the accuracy of the proposed SEP approximation as well as the performance of the proposed SEP upper bounds. We examine R-HQAM constellations and we emphasize on higher order HQAM constellations.

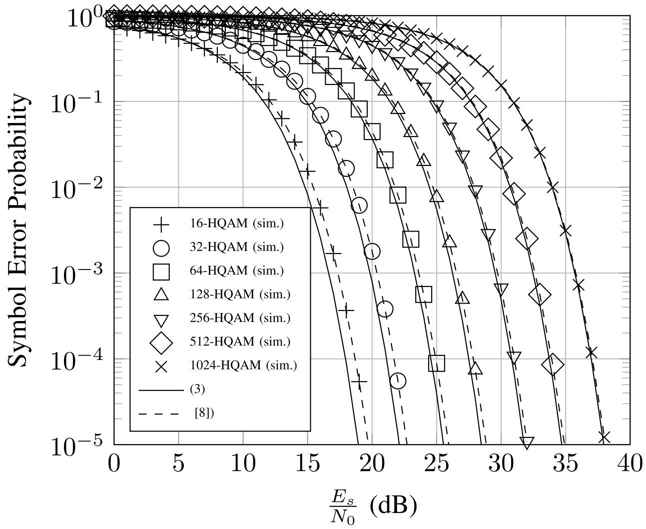

Fig. 3. SEP versus received SNR.

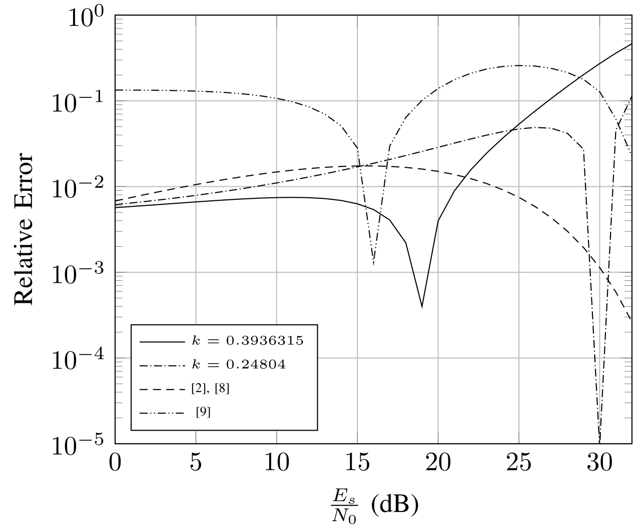

Fig. 4. 256-HQAM relative error vs received SNR.

Fig. 3 compares the SEP approximation in (3) with the simulated SEP and the SEP approximation of [8] for regular HQAM constellations of different orders. It should be mentioned that the approximation method presented in [2] is the same as in [8]. In all cases, the accuracy of the proposed approximation method is verified. More specifically, as the order of the constellation increases, the proposed SEP approximation becomes tighter, as illustrated in Fig. 3, which verifies the contribution of our method especially in higher order HQAM constellations. This can be justified by taking into account that in higher order HQAM constellations, e.g., $M \geq 512$, most of the decision regions are either hexagons or infinite striped-shaped areas which can be tightly approximated by (3).

Fig. 4 illustrates the relative error on the exact SEP of 256-HQAM constellation for the proposed method as well as for the methods presented in [2], [8] and [9], respectively. Specifically, the relative error is a measure of the uncertainty of the approximation compared to the exact SEP value. As it can be observed, the proposed method for the value of k presented in Table II outperforms the methods described in the existing literature in low and medium SNR regime. Furthermore, in the high SNR regime the methods described in [8] and [9] perform better than the proposed one. However, by changing the value of k the approximation in high SNR regime can be further improved while its tightness will deteriorate in the low and medium SNR regime. More specifically, by choosing an SNR value and minimizing the squared error of the proposed approximation from the exact SEP for the specific SNR value, a different k can be calculated. For instance, as shown in Fig. 4, by choosing SNR $= 30$ dB and applying the aforementioned minimization problem the value $k = 0.24804$ is obtained.

In Fig. 5, we compare the upper bounds given by Remark 2 and Remark 3 highlighting their tightness compared with the simulated SEP. In 16-HQAM, where the number of the external symbols are comparable to the one of the internal symbols, the improved upper bound for the SEP given by Remark 3 leads to a tighter bound compared to the one in Remark 2. However, the proposed upper bound in Remark 3 remains very useful due to its simple mathematical expression. This is highlighted on the higher order constellations where it can be observed that the two upper bounds display the same performance. Specifically, it becomes apparent that as the order of the constellation increases the difference between the two bounds is negligible, since the ratio of external symbols to internal symbols decreases.

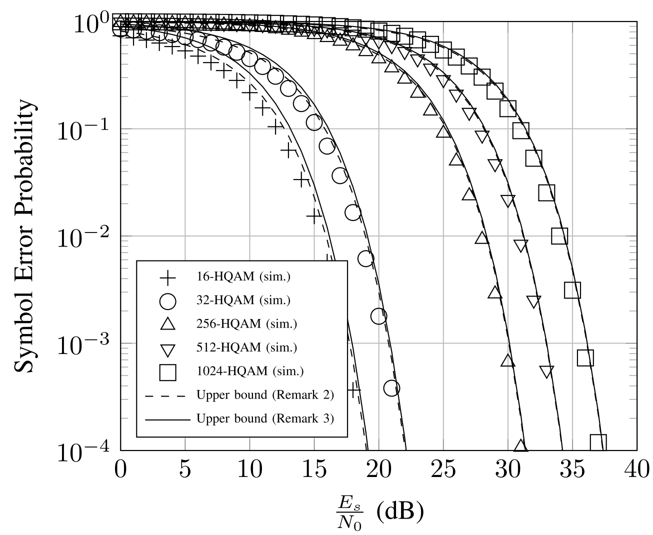

Fig. 5. Upper bounds versus received SNR.

Finally, in Fig. 6, we illustrate the detection efficiency of the proposed detection scheme compared to the regular MLD method. It is evident that the detection errors of the proposed detection method is equivalent to MLD. As a result, we conclude that the proposed detection algorithm provides high detection efficiency while decreases the time complexity from $\mathcal{O}(M)$ to $\mathcal{O}(\log \sqrt{M})$.

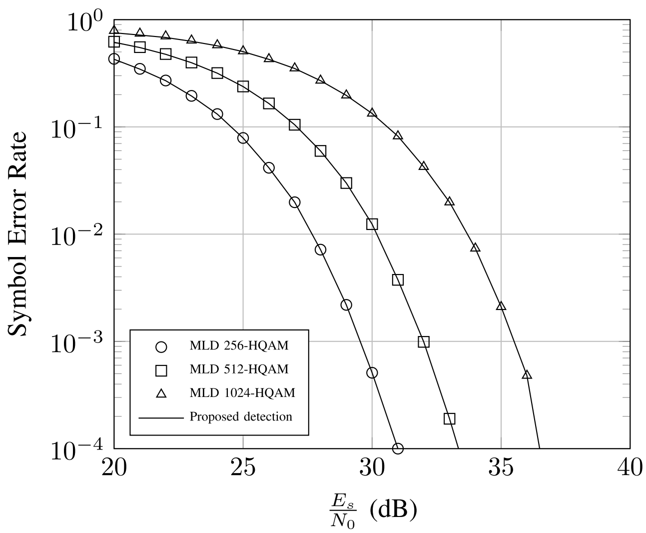

Fig. 6. Detected symbol error rate versus received SNR.

VI. Conclusion

In this work, we have proposed a simple and accurate closed-form SEP approximation for HQAM constellations as well as two simple and tight upper bounds. Specifically, the proposed SEP approximation is simple and accurate for all HQAM constellations, especially for higher order constellations ($M \geq 256$). Finally, we have proposed a detection scheme of average complexity $\mathcal{O}(\log\sqrt{M})$ which provides detection efficiency equivalent to the MLD.

Appendix A
Proof of Theorem 1

The probability of the received symbol r lying into the circular area can be calculated as

$$P_c = \Pr\left(n_x^2 + n_y^2 \leq \rho^2\right), \tag{11}$$

where $\Pr(\cdot)$ denotes probability and n_x and n_y denote the horizontal and vertical component of the RV n, respectively. Moreover, the RVs n_x and n_y are independent and follow normal distribution, i.e., $n_x, n_y \sim \mathcal{N}(0, \frac{N_0}{2})$. Furthermore, the sum of squares of two independent and identically distributed RVs, that follow normal distribution $\mathcal{N}(0, \frac{N_0}{2})$, follows the exponential distribution with rate parameter $\frac{1}{N_0}$. Thus, P_c can be calculated as

$$P_c = 1 - e^{-\frac{\rho^2}{N_0}}. \tag{12}$$

The decision region of the internal symbols of an HQAM constellation can be approximated by a circle, since their decision region is a regular hexagon. However, the decision region of the external symbols is infinite. Therefore, instead of approximating the infinite decision region of an external symbol with a circle, it can be approximated by a semicircle with radius ρ and an infinite rectangle tangential to the semicircle, as illustrated in Fig. 1. Thus, in order to derive the SEP approximation, the decision region of the external symbols can be approximated with a region $\hat{A} \in [A, A']$, where A and A' are semicircle areas with radii R and R', respectively, and a region B which is the infinite rectangle tangential to region $\hat{A}$, as it is illustrated in Fig. 1. As a result, the probability that the received signal r lies inside region $\hat{A} + B$ for an AWGN channel is given by

$$P(\hat{A} \cup B) = P(\hat{A}) + P(B), \tag{13}$$

where

$$P(\hat{A}) = \frac{1}{2}\left(1 - e^{-\frac{\rho^2}{N_0}}\right) \tag{14}$$

and

$$P(B) = \frac{1}{2}\left(1 - 2Q\left(\frac{\rho\sqrt{2}}{\sqrt{N_0}}\right)\right). \tag{15}$$

Thus, the erroneous detection probability of external symbols is given by

$$P_e = 1 - P(\hat{A}) - P(B). \tag{16}$$

Using (12) and (16), the constellation's SEP can be approximated as

$$P_s = \frac{M-b}{M}(1 - P_c) + \frac{b}{M}P_e. \tag{17}$$

After some algebraic manipulations, (3) is derived which concludes the proof.

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