Transactions Letters

Closed-Form Statistics for the Sum of Squared Nakagami-m Variates and Its Applications

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***Abstract*—We present closed-form expressions for the probability density function (PDF) and the cumulative distribution function (CDF) of the sum of non-identical squared Nakagami-m random variables (RVs) with integer-order fading parameters. As it is shown, they can be written as a weighted sum of Erlang PDFs and CDFs, respectively, while the analysis includes both independent and correlated sums of RVs. The proposed formulation significantly improves previously published results, which are either in the form of infinite sums or higher order derivatives of the fading parameter m. The obtained formulas can be applied to the performance analysis of diversity combining receivers operating over Nakagami-m fading channels.**

***Index Terms*—Average symbol-error probability (ASEP), diversity, maximal ratio combining (MRC), Nakagami-m fading, outage probability, Shannon's channel capacity, sum of Erlang variates.**

I. Introduction

PERFORMANCE analysis of digital wireless communications systems usually deals with complicated and cumbersome statistical tasks. One of them arises in the study of diversity combining receivers operating over Nakagami-m fading channels [2], where the statistics of the sum of squared Nakagami-m random variables (RVs) (or equivalently, the sum of Gamma RVs) is required. Well-known applications in the field of wireless communication systems, where such sums can be useful, are maximal ratio combining (MRC) and postdetection equal gain combining (EGC). Moreover, they can be used for the evaluation of the outage probability in cellular systems with cochannel interference (CCI) (see [3]–[9] and references therein).

The most general approach related to the distribution of the sum of Gamma RVs has been presented by Moschopoulos in [10], where an infinite-series representation for the probability density function (PDF) of the sum of independent nonidentical Gamma RVs has been proposed. Alouini *et al.* in [9] have extended the result of [10] for the case of arbitrarily correlated Gamma RVs, and studied the performance of MRC and postdetection EGC receivers, as well as receivers in the presence of CCI. However, in [3]–[8], by following the moment generating function-based approach [11, Sec. 9] or starting from the characteristic function, the performance analytical formulas derived (e.g., for the average symbol-error probability (ASEP) and the outage probability) are in the form of either infinite sums or higher order derivatives of the fading parameter. This occurs since, to the best of the authors' knowledge, there are no simple closed-form expressions either for the PDF or the cumulative distribution function (CDF) of the sum of nonidentically distributed Gamma RVs available in the open technical literature.[1]

In this letter, novel closed-form expressions for the PDF and the CDF of the sum of nonidentical squared Nakagami-m RVs, with integer-order fading parameters, are derived. Our analysis is not only limited to independent fading, but correlated fading channels are also included. Furthermore, in order to reveal the importance of the proposed statistical formulation and by following the PDF-based approach, we study the performance of L-branch MRC and postdetection squared-law combining (SLC) receivers in the presence of Nakagami-m multipath fading. For these applications, closed-form expressions for the outage probability, the channel average spectral efficiency (SE), and the ASEP for several coherent, noncoherent, binary, and multilevel modulation signalings are obtained.

After this short introduction, in Section II, novel closed-form expressions for the PDF and the CDF of the sum of squared Nakagami-m RVs are obtained. In Section III, the theoretical results of Section II are applied to derive useful expressions for various performance metrics of diversity receivers, operating over Nakagami-m fading channels. Finally, in Section IV, useful concluding remarks are provided.

II. Closed-Form Statistics for The Sum of Squared Nakagami-m RVs

Let $\{X_\ell\}_{\ell=1}^{L}$ be L Nakagami-m distributed RVs, with PDF given by[2] [2]

$$f_{X_\ell}(x; m_\ell, \eta_\ell) = \frac{2x^{2m_\ell-1}}{\eta_\ell^{m_\ell}(m_\ell-1)!}\exp\left(-\frac{x^2}{\eta_\ell}\right)\mathrm{U}(x) \quad (1)$$

[1]Two independent and parallel works are [12] and [13].

[2]Note that in the case where the fading parameter m_ℓ is a positive integer, (1) is an alternative form of the classical Nakagami-m PDF [2].

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where $\mathrm{U}(x)$ is the well-known unit step function defined as $\mathrm{U}(x \geq 0) = 1$ and zero otherwise, m_ℓ denotes the Nakagami-m fading parameter, here considered as a positive integer, and $\eta_\ell = E\langle X_\ell^2\rangle / m_\ell$, with $E\langle\cdot\rangle$ denoting expectation. Moreover, the squared value of X_ℓ, $Y_\ell = X_\ell^2$, follows the Erlang distribution[3] with PDF given by

$$f_{Y_\ell}(x; m_\ell, \eta_\ell) = \frac{x^{m_\ell - 1}}{\eta_\ell^{m_\ell}(m_\ell - 1)!} \exp\left(-\frac{x}{\eta_\ell}\right) \mathrm{U}(x) \quad (2)$$

and CDF [11]

$$F_{Y_\ell}(x; m_\ell, \eta_\ell) = \left[1 - \frac{\Gamma(m_\ell, x/\eta_\ell)}{(m_\ell - 1)!}\right] \mathrm{U}(x) \quad (3)$$

where $\Gamma(\cdot,\cdot)$ is the incomplete Gamma function defined in [14, eq. (8.350.2)]. Using [14, eq. (8.352.2)], the above CDF can be rewritten as

$$F_{Y_\ell}(x; m_\ell, \eta_\ell) = \left[1 - \exp\left(-\frac{x}{\eta_\ell}\right) \sum_{\mu=1}^{m_\ell - 1} \frac{1}{\mu}\left(\frac{x}{\eta_\ell}\right)^{\mu}\right] \mathrm{U}(x). \quad (4)$$

A. Independent RVs

Theorem 1: (PDF of the Sum of Squared Nakagami-m RVs): Let $\{Y_\ell\}_{\ell=1}^{L}$ be a set of RVs following the PDF defined in (2), with $\eta_i \neq \eta_j \ \forall i \neq j$. Then, the PDF of the sum

$$Z_L = \sum_{i=1}^{L} Y_i \quad (5)$$

is a nested finite weighted sum of Erlang PDFs, given by

$$f_{Z_L}(z) = \sum_{i=1}^{L}\sum_{k=1}^{m_i} \Xi_L\left(i, k, \{m_q\}_{q=1}^{L}, \{\eta_q\}_{q=1}^{L}, \{l_q\}_{q=1}^{L-2}\right) \times f_{Y_i}(z; k, \eta_i) \quad (6)$$

where the weights Ξ_L are shown in (7) and R_L is defined as $R_L \triangleq \sum_{i=1}^{L} m_i$.

Proof: The proof is given in the Appendix. ■

For the special case where $m_\ell = 1 \ \forall \ell$ (i.e., Rayleigh fading), it can be easily verified that (6) reduces to [15, eq. (10)]. Note that in order to quickly and easily evaluate Ξ_L, we have managed to develop a recursive formula given by (8)

$$\begin{aligned}
\Xi_L&\left(i, k, \{m_q\}_{q=1}^{L}, \{\eta_q\}_{q=1}^{L}, \{l_q\}_{q=1}^{L-2}\right) \\
&= \sum_{l_1=k}^{m_i}\sum_{l_2=k}^{l_1}\cdots\sum_{l_{L-2}=k}^{l_{L-3}} \left[\frac{(-1)^{R_L - m_i}\eta_i^k}{\prod_{h=1}^{L}\eta_h^{m_h}}\right. \\
&\times \frac{\left(m_i + m_{1+\mathrm{U}(1-i)} - l_1 - 1\right)!}{\left(m_{1+\mathrm{U}(1-i)} - 1\right)!\,(m_i - l_1)!} \\
&\times \left(\frac{1}{\eta_i} - \frac{1}{\eta_{1+\mathrm{U}(1-i)}}\right)^{l_1 - m_i - m_{1+\mathrm{U}(1-i)}} \\
&\times \frac{\left(l_{L-2} + m_{L-1+\mathrm{U}(L-1-i)} - k - 1\right)!}{\left(m_{L-1+\mathrm{U}(L-1-i)} - 1\right)!(l_{L-2} - k)!} \\
&\times \left(\frac{1}{\eta_i} - \frac{1}{\eta_{L-1+\mathrm{U}(L-1-i)}}\right)^{k - l_{L-2} - m_{L-1+\mathrm{U}(L-1-i)}} \\
&\times \prod_{s=1}^{L-3} \frac{\left(l_s + m_{s+1+\mathrm{U}(s+1-i)} - l_{s+1} - 1\right)!}{\left(m_{s+1+\mathrm{U}(s+1-i)} - 1\right)!\,(l_s - l_{s+1})!} \\
&\left.\times \left(\frac{1}{\eta_i} - \frac{1}{\eta_{s+1+\mathrm{U}(s+1-i)}}\right)^{l_{s+1} - l_s - m_{s+1+\mathrm{U}(s+1-i)}}\right]
\end{aligned} \quad (7)$$

$$\begin{aligned}
\Xi_L&\left(i, m_i - k, \{m_q\}_{q=1}^{L}, \{\eta_q\}_{q=1}^{L}, \{l_q\}_{q=1}^{L-2}\right) \\
&= \frac{1}{k}\sum_{\substack{j,q=1 \\ q\neq i}}^{L} \frac{m_q}{\eta_i^j}\left(\frac{1}{\eta_i} - \frac{1}{\eta_q}\right)^{-j} \\
&\times \Xi_L\left(i, m_i - k + j, \{m_q\}_{q=1}^{L}, \{\eta_q\}_{q=1}^{L}, \{l_q\}_{q=1}^{L-2}\right)
\end{aligned} \quad (8)$$

with

$$\Xi_L\left(i, m_i, \{m_q\}_{q=1}^{L}, \{\eta_q\}_{q=1}^{L}, \{l_q\}_{q=1}^{L-2}\right) = \frac{\eta_i^{m_i}}{\prod_{h=1}^{L}\eta_h^{m_h}} \prod_{\substack{j=1 \\ j\neq i}}^{L} \left(\frac{1}{\eta_j} - \frac{1}{\eta_i}\right)^{-m_j}.$$

Corollary 1 (CDF of the Sum of Squared Nakagami-m RVs): The CDF of Z_L is given by

$$F_{Z_L}(z) = \sum_{i=1}^{L}\sum_{k=1}^{m_i} \Xi_L\left(i, k, \{m_q\}_{q=1}^{L}, \{\eta_q\}_{q=1}^{L}, \{l_q\}_{q=1}^{L-2}\right) \times F_{Y_i}(z; k, \eta_i). \quad (9)$$

Proof: The CDF of Z_L can be easily obtained by integrating (6) from 0 to z and interchanging the order of summations and integrations. ■

To the best of the authors' knowledge, (6) and (9) are novel. Interestingly enough, both expressions can be easily evaluated due to the fact that only simple elementary functions (i.e., powers and exponentials) are included. Moreover, (6) is simpler, compared with the corresponding PDF expression presented in [3, eq. (10)], which is apparently not in closed form, since it includes higher order derivatives as functions of the parameter m. Note also that by using [3, eq. (10)], it seems to be difficult, if not impossible, to study other important metrics, such as CDF.

B. Correlated RVs

In order to obtain the sum of correlated squared Nakagami-m RVs, the following assumptions, made also in [16]–[18], are taken into account and repeated here for the reader's convenience.

1) Without loss of generality, it can be assumed that statistical parameters m_ℓ are in increasing order, i.e., $m_1 \leq m_2 \leq \cdots \leq m_L$.
2) Let $\{X_\ell\}$ be arbitrarily correlated Nakagami-m RVs with marginal PDFs given by (1).
3) Let $\mathbf{W}_\ell$ be $2m_\ell \times 1$-dimensional vectors defined as $\mathbf{W}_\ell = [W_{\ell,1}W_{\ell,2}\cdots W_{\ell,2m_\ell}]^\dagger$, where $(\cdot)^\dagger$ denotes transpose, and the elements $\{W_{\ell,k}\}_{k=1}^{2m_\ell}$ are independent and

[3]The Erlang distribution is a special case of the well-known Gamma distribution for integer values of m_ℓ.

identically distributed zero-mean Gaussian RVs with variance $E\langle W_{\ell,k}^2\rangle = \eta_\ell/2$.

4) Let $\mathbf{W}$ be a vector $D_T \times 1$ order, defined as $\mathbf{W} = [\mathbf{W}_1^\dagger \mathbf{W}_2^\dagger \cdots \mathbf{W}_L^\dagger]^\dagger$, where $D_T = \sum_{i=1}^N 2m_i$, with covariance matrix given by $\mathcal{K}_W = E\langle \mathbf{W}\mathbf{W}^\dagger\rangle$.

5) Without loss of generality, let the correlation among the elements of $\mathbf{W}$ be constructed such that

$$E\langle W_{i,k}W_{j,l}\rangle = \begin{cases} \eta_i/2, & \text{if } i=j \text{ and } k=l \\ \rho_{i,j}\sqrt{\eta_i\eta_j}/2, & \text{if } i\neq j \text{ and } \\ & k=l=1,2, \\ & \ldots, 2\min\{m_i,m_j\} \\ 0, & \text{otherwise.} \end{cases}$$

It can be shown that the relation between the covariance of Y_i, Y_j and the correlation of the elements of $\mathbf{W}$ is given by

$$\rho_{Y_i,Y_j} = \frac{E\langle (Y_i - m_i\eta_i)(Y_j - m_j\eta_j)\rangle}{\sqrt{\mathrm{var}(Y_i)\mathrm{var}(Y_j)}} = \frac{\min\{m_i,m_j\}}{\sqrt{m_i m_j}}\rho_{i,j}^2 \tag{10}$$

where $\mathrm{var}(Y_\ell) = \eta_\ell^2 m_\ell$.

6) Let $\{\lambda_\ell\}$ be the set of L distinct eigenvalues of $\mathcal{K}_W$, where each λ_ℓ has algebraic multiplicity μ_ℓ, such that $\sum_{i=1}^L \mu_i = D_T$.

Theorem 2: (PDF of the Sum of Squared Correlated Nakagami-m RVs): Let

$$Z_L = \sum_{i=1}^L X_i^2 \tag{11}$$

then it holds that

$$Z_L \overset{d}{=} \sum_{i=1}^L V_i \tag{12}$$

where V_ℓ is the ℓth Erlang distributed RV with parameters $m_\ell = \mu_\ell/2, \eta_\ell = 4\lambda_\ell/\mu_\ell$ and the notation "$\overset{d}{=}$" means "equality in distribution."

Proof: See [16], where the Karhunen–Loeve expansion is used to decorrelate arbitrarily correlated, nonidentical, Erlang distributed RVs. ■

Lemma 1: (CDF of the Sum of Squared Correlated Nakagami-m RVs): The CDF of the sum of arbitrarily correlated squared Nakagami-m RVs can be found in closed form using *Corollary 1* and *Theorem 2*.

III. Diversity Combining Receivers

We consider an L-branch diversity receiver operating over a multipath fading environment. The baseband received signal at the ℓth, $\ell = 1, 2, \ldots, L$, diversity branch is

$$\zeta_\ell = sX_\ell + n_\ell \tag{13}$$

where s is the transmitted symbol, with energy $E_s = E\langle |s|^2\rangle, X_\ell$ is the Nakagami-m distributed fading envelope, and n_ℓ is the additive white Gaussian noise, with a single-sided power spectral density N_0. The noise components are assumed to be statistically independent of the signal and uncorrelated with each other. Moreover, the channel is considered as being slowly time-varying, and thus, its characteristics are perfectly known to the receiver.

The instantaneous signal-to-noise ratio (SNR) per symbol $\gamma_\ell = X_\ell^2 E_s/N_0$ in the ℓth input branch follows the Erlang distribution $f_{Y_\ell}(\gamma_\ell; m_\ell, \overline{\gamma}_\ell/m_\ell)$, with m_ℓ and $\overline{\gamma}_\ell = E\langle X_\ell^2\rangle E_s/N_0$ being the corresponding Nakagami-m fading parameter and the average input SNR per symbol, respectively. The performance analysis of the MRC and postdetection SLC receivers [19], in which the instantaneous SNR per symbol at the output is given by the well-known expression $\gamma = \sum_{i=1}^L \gamma_i$, can be tackled using the analysis presented in Section II for both independent and correlative fading.

A. Maximal Ratio Diversity

1) Outage Probability: The outage probability in noise-limited systems, P_{out}, is defined as the probability that the instantaneous MRC output SNR falls below a given outage threshold γ_{th}. This probability can be easily obtained by replacing z with γ_{th} in (9) as

$$P_{\mathrm{out}}(\gamma_{\mathrm{th}}) = F_{Z_L}(\gamma_{\mathrm{th}}) \tag{14}$$

with $\eta_q = \overline{\gamma}_q/m_q$, for the independent case, or using *Lemma 1* for the correlative case. Note that our approach can be efficiently applied to evaluate the outage probability in cellular systems, in which CCI may be further assumed.

In Fig. 1, an MRC receiver with $L = 4$ antennae, operating in a Nakagami-m multipath fading environment, with $m_1 = m_2 = 1, m_3 = 2$, and $m_4 = 4$, is considered. Moreover, an exponentially decaying power delay profile (PDP) $\overline{\gamma}_\ell = \overline{\gamma}_1 \exp[-\delta(\ell-1)]$ is assumed with power-decaying factors $\delta = 0.5, 1$, and exponential correlation[4] among the input channels $\rho_{\gamma_i,\gamma_j} = \rho^{|i-j|}$ with $\rho = 0.3, 0.7$ $(i,j = 1,2,3,4)$. Note that the correlation matrix of this model corresponds to the scenario of multichannel reception from equispaced diversity antennae [4]. In this figure, P_{out} is plotted as a function of the inverse, normalized to $\overline{\gamma}_1$, outage threshold $\overline{\gamma}_1/\gamma_{\mathrm{th}}$. The obtained results, which have been also verified by Monte Carlo simulations, clearly show that the outage performance degrades with an increase of the fading correlation and/or the power-decay factor.

2) Shannon Channel Capacity: It is well known that the Shannon channel capacity provides an upper bound of maximum transmission rate in a given Gaussian environment. The average SE, in Shannon's sense, defined as the normalized (by the transmitted signal's bandwidth) average channel capacity, is given by [21]

$$\overline{S}_e = \int_0^\infty \log_2(1+\gamma) f_{Z_L}(\gamma)\, d\gamma. \tag{15}$$

[4]For $\rho \to 1$, it can be verified that the covariance matrix $\mathcal{K}_W$ is not a positive definite matrix (i.e., some eigenvalues are complex or not positive). Therefore, it is not possible to study cases for values of ρ close to 1. This fact can be also explained since two Nakagami RVs with different distribution parameters can not be completely correlated [20].

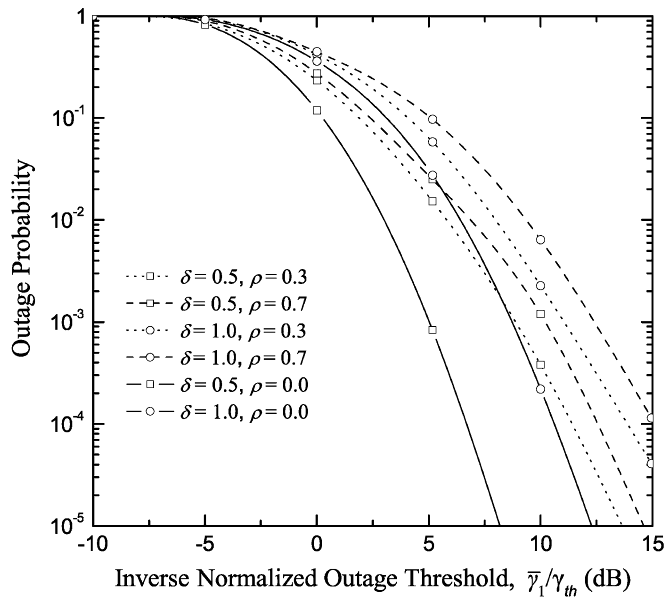

Fig. 1. Outage probability versus $\overline{\gamma}_1/\gamma_{\rm th}$ for $L = 4$, with an exponentially input decaying PDP and fading parameters $m_1 = m_2 = 1, m_3 = 2$, and $m_4 = 4$.

By substituting (6) in the above integral and using [14, eq. (4.358.1)] and [22, eq. (06.06.20.0001.01)], in case of independent fading, the average channel SE can be written in closed form as in (16), where $\mathcal{C}$ is the Euler's constant [14, Sec. 9.73] ${}_2F_2(\cdot,\cdot;\cdot,\cdot;\cdot)$ is a generalized hypergeometric series [14, eq. (9.14.1)], and

$$\binom{k-1}{w-1} = \frac{(k-1)!}{[(w-1)!\,(k-w)!]}.$$

In case of correlative fading, the average SE can be obtained using (16) and substituting m_i with $\mu_i/2$, and $\overline{\gamma}_i$ with $\lambda_i/2$

$$\overline{S}_e = \frac{1}{\ln 2}\sum_{i=1}^{L}\sum_{k=1}^{m_i} \times \Xi_L\left(i,k,\{m_q\}_{q=1}^{L},\left\{\frac{\overline{\gamma}_q}{m_q}\right\}_{q=1}^{L},\{l_q\}_{q=1}^{L-2}\right) \times \frac{1}{(k-1)!}\left(\frac{m_i}{\overline{\gamma}_i}\right)^k \sum_{w=1}^{k}\binom{k-1}{w-1}(-1)^{k-w} \times \exp\left(\frac{m_i}{\overline{\gamma}_i}\right)\left\{\frac{1}{w^2}\,{}_2F_2\left(w,w;w+1,w+1;-\frac{m_i}{\overline{\gamma}_i}\right) + \left(\frac{\overline{\gamma}_i}{m_i}\right)^w (w-1)!\left[\ln\left(\frac{\overline{\gamma}_i}{m_i}\right)+\sum_{h=1}^{w-1}\frac{1}{h}-\mathcal{C}\right]\right\}. \tag{16}$$

3) Error Performance: The most straightforward approach to obtain the ASEP $\overline{P}_{\rm se}$ is to average the conditional SEP $P_{\rm se}(\gamma)$ over the PDF of the combiner output SNR [11], i.e.,

$$\overline{P}_{\rm se} = \int_0^\infty P_{\rm se}(\gamma) f_{Z_L}(\gamma)d\gamma. \tag{17}$$

It is well known that for several signaling constellations, $P_{\rm se}(\gamma)$ can be written as follows.

TABLE I
PARAMETERS A AND B FOR SEVERAL SIGNALING CONSTELLATIONS

Modulation Scheme	A	B	M
BPSK	1/2	1	–
BFSK	1/2	1/2	–
GMSK	1	$\mathcal{B}$	–
M-DEPSK	2	$\sin^2(\pi/M)$	≥ 2
QPSK	1	1/2	–
M-PSK	1	$\sin^2(\pi/M)$	> 4
M-FSK	$(M$-1)/2	1/2	> 2
Square M-QAM	$2-2/\sqrt{M}$	$\frac{3}{2(M-1)}$	≥ 4
M-DPSK	1	$2\sin^2[\pi/(2M)]$	≥ 2
DBPSK	1/2	1	–
M-NFSK	$\sum_{k=1}^{M-1}\frac{(-1)^{k+1}}{k+1}\binom{M-1}{k}$	$\frac{k}{k+1}$	≥ 2

1) For binary phase-shift keying (BPSK), binary frequency-shift keying (BFSK), and for high values of average input SNR for Gaussian minimum-shift keying (GMSK),[5] M-ary-differentially encoded phase-shift keying (M-DEPSK), quadrature phase-shift keying (QPSK), M-ary phase-shift keying (M-PSK), M-ary frequency-shift keying (M-FSK), square M-ary quadrature amplitude modulation (M-QAM), and M-ary-differential PSK (M-DPSK) in the form of $P_{\rm se}(\gamma) = A{\rm erfc}(\sqrt{B\gamma})$, where ${\rm erfc}(\cdot)$ is the complementary error function [14, eq. (8.250.4)].
2) For differential binary PSK (DBPSK) and M-ary noncoherent frequency-shift keying (M-NFSK), in the form of $P_{\rm se}(\gamma) = A\exp(-B\gamma)$.

The particular values of A and B depend on the considered modulation scheme and are summarized in Table I. In the following, $\overline{P}_{\rm se}$ will be obtained in closed form for each of the above two cases.

By substituting (6) in (17), it can be easily recognized that for coherent binary and M-ary modulation schemes, such as: 1) BPSK and BFSK; and 2) for high values of the average input SNR for GMSK, M-DEPSK, QPSK, M-PSK, M-FSK, M-QAM, and M-DPSK, the evaluation of integrals of the form $\Upsilon = \int_0^\infty x^{k-1}{\rm erfc}(\sqrt{Bx})\exp(-x/\eta_i)dx$ with $\eta_i = \overline{\gamma}_i/m_i$ is required. The above integral can be evaluated via [14, eq. (6.455.1)] by noting that ${\rm erfc}(\cdot)$ can be expressed as an incomplete Gamma function [22, eq. (06.06.03.0004.01)]. Therefore, the ASEP can be derived in closed form as

$$\overline{P}_{\rm se} = A\sum_{i=1}^{L}\sum_{k=1}^{m_i} \times \Xi_L\left(i,k,\{m_q\}_{q=1}^{L},\left\{\frac{\overline{\gamma}_q}{m_q}\right\}_{q=1}^{L},\{l_q\}_{q=1}^{L-2}\right) \times \frac{(2k-1)!!}{k!(2B)^k}\left(\frac{m_i}{\overline{\gamma}_i}\right)^k \times {}_2F_1\left(k,k+\frac{1}{2};k+1;-\frac{m_i}{B\overline{\gamma}_i}\right) \tag{18}$$

[5] $\mathcal{B}$ is determined by the bandwidth of the premodulation Gaussian filter.

where ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the Gauss hypergeometric function [14, eq. (9.100)].

The ASEP of noncoherent modulation schemes, such as M-NFSK and DBPSK, can be extracted by substituting (6) in (17) and using [14, eq. (3.381.4)], yielding

$$\overline{P}_{\rm se} = A\sum_{i=1}^{L}\sum_{k=1}^{m_i} \times \Xi_L\left(i,k,\{m_q\}_{q=1}^{L},\left\{\frac{\overline{\gamma}_q}{m_q}\right\}_{q=1}^{L},\{l_q\}_{q=1}^{L-2}\right) \times\left(\frac{m_i}{\overline{\gamma}_i}\right)^{k}\left(B+\frac{m_i}{\overline{\gamma}_i}\right)^{-k}. \tag{19}$$

For correlative fading, the ASEP can be obtained using (18) and (19), after substituting m_i with $\mu_i/2$ and $\overline{\gamma}_i$ with $\lambda_i/2$.

B. Postdetection Diversity With M-FSK

In postdetection SLC with M-FSK signaling, the receiver consists of L receivers, where each one of them has M branches, each followed by a tunable bandpass filter, which is followed by a square-law detector. The outputs of the L receivers, corresponding to each received signal, are combined to form the M decision variables, and it has been proved that the conditional SEP is [19, eq. (9)]

$$P_{\rm se}(\gamma) = \frac{1}{(L-1)!}\sum_{n=1}^{M-1}\sum_{q=1}^{(L-1)n}\sigma_{q,n}\binom{M-1}{n}(-1)^{n+1} \times\frac{(q+L-1)!}{(n+1)^{q+L}}\exp(-\gamma)\,{}_1F_1\left(q+L;L;\frac{\gamma}{n+1}\right) \tag{20}$$

$$\overline{P}_{\rm se} = \frac{1}{(L-1)!}\sum_{i=1}^{L}\sum_{k=1}^{m_i} \times\Xi_L\left(i,k,\{m_q\}_{q=1}^{L},\left\{\frac{\overline{\gamma}_q}{m_q}\right\}_{q=1}^{L},\{l_q\}_{q=1}^{L-2}\right)\left(\frac{m_i}{\overline{\gamma}_i}\right)^{k} \times\sum_{n=1}^{M-1}\sum_{q=0}^{(L-1)n}\sigma_{q,n}\binom{M-1}{n}(-1)^{n+1} \times\frac{(q+L-1)!}{(n+1)^{q+L}}\left(1+\frac{m_i}{\overline{\gamma}_i}\right)^{-k}{}_2F_1 \times\left(q+L,k;L;\frac{1}{(n+1)\left(1+m_i/\overline{\gamma}_i\right)}\right) \tag{21}$$

where $\sigma_{q,n}=1$ for $q=0, \sigma_{q,n}=\sigma_{q-r,n}\sum_{r=1}^{k}[r(n+1)/q-1]/r!$ for $r\le L-1$, and zero, otherwise. For SLC receivers with M-FSK signaling and noncoherent detection, by averaging (20) over (17) and using [14, eq. (7.621.4)], the ASEP can be obtained in closed form as in (21). (21) improves on previously published results [19], where the corresponding expressions include integrals with infinite limits. For correlated fading channels, (21) also holds after substituting m_i with $\mu_i/2$ and $\overline{\gamma}_i$ with $\lambda_i/2$.

IV. Conclusion

We derived novel closed-form expressions for the PDF and the CDF of the sum of squared, nonidentical, independent or correlated Nakagami-m RVs in the case of integer-order Nakagami-m fading parameters. An interesting finding is that these expressions can be written as a weighted sum of Erlang distributions. Based on the statistical formulas obtained, and following the PDF-based approach, MRC and postdetection M-FSK SLC receivers were studied, and important performance metrics, such as outage probability, average SE, and ASEP, were expressed in closed form. Our results improve on previously published ones, which are either in the form of infinite sums or higher order derivatives of the fading parameter.

Appendix
Proof of *Theorem 1*

In order to derive the PDF of Z_L [see (5)] in closed form, we follow three steps.

1) Step 1 ($L=2$ Terms): For $L=2$, the PDF of $Z_2 = Y_1+Y_2$ can be evaluated as

$$f_{Z_2}(z) = \int_0^z f_{Y_1}(x;m_1,\eta_1)f_{Y_2}(z-x;m_2,\eta_2)\,dx \tag{A-1}$$

in which, by substituting (2) and using [14, eq. (3.383.1)], can be expressed as[6]

$$f_{Z_2}(z) = \frac{{\rm B}(m_2,m_1)}{\eta_1^{m_1}\eta_2^{m_2}\Gamma(m_1)\Gamma(m_2)}z^{m_1+m_2-1}\exp\left(-\frac{z}{\eta_2}\right) \times{}_1F_1\left(m_1;m_1+m_2;-\left(\frac{1}{\eta_1}-\frac{1}{\eta_2}\right)z\right){\rm U}(z) \tag{A-2}$$

$$f_{Z_2}(z) = \frac{(1-m_1-m_2)_{m_1}\left(\frac{1}{\eta_2}-\frac{1}{\eta_1}\right)^{1-m_1-m_2}}{\eta_1^{m_1}\eta_2^{m_2}(m_1+m_2-1)(m_1-1)!} \times\exp\left(-\frac{z}{\eta_2}\right)\left\{\sum_{k=0}^{m_2-1}\frac{(1-m_2)_k\left[-\left(\frac{1}{\eta_1}-\frac{1}{\eta_2}\right)z\right]^k}{k!(2-m_1-m_2)_k} - \exp\left[-\left(\frac{1}{\eta_1}-\frac{1}{\eta_2}\right)z\right] \times\sum_{k=0}^{m_1-1}\frac{(1-m_1)_k\left[\left(\frac{1}{\eta_1}-\frac{1}{\eta_2}\right)z\right]^k}{k!(2-m_1-m_2)_k}\right\}{\rm U}(z) \tag{A-3}$$

where ${\rm B}(\cdot,\cdot)$ is the Euler Beta function [14, eq. (8.380.1)], and ${}_1F_1(\cdot;\cdot;\cdot)$ is the Kummer confluent hypergeometric function [14, eq. (9.210.1)]. For m_ℓ being positive integers and by using [22, eq. (07.20.03.0024.01)], (A-2) can be rewritten as shown in (A-3), with $(n)_k = \Gamma(n+k)/\Gamma(k)$ being the Pochammer

[6]Note that (A-2) is a generic result concerning the PDF of the sum of two Gamma RVs for arbitrary (not necessarily integer) values of m_ℓ.

$$\Xi_2(i,k,m_1,m_2,\eta_1,\eta_2) = (-1)^{R_2-m_i}\frac{\eta_i^k(m_1+m_2-k-1)!\left(\frac{1}{\eta_i}-\frac{1}{\eta_{1+\mathrm{U}(1-i)}}\right)^{k-m_1-m_2}}{\eta_1^{m_1}\eta_2^{m_2}(m_{1+\mathrm{U}(1-i)}-1)!(m_i-k)!} \tag{A-5}$$

$$\Xi_3(i,k,m_1,m_2,m_3,\eta_1,\eta_2,\eta_3,l_1) = \sum_{l_1=k}^{m_i}(-1)^{R_3-m_i}\frac{\eta_i^k(m_i+m_{1+\mathrm{U}(1-i)}-l_1-1)!\left(\frac{1}{\eta_i}-\frac{1}{\eta_{1+\mathrm{U}(1-i)}}\right)^{l_1-m_i-m_{1+\mathrm{U}(1-i)}}}{\eta_1^{m_1}\eta_2^{m_2}\eta_3^{m_3}(m_{1+\mathrm{U}(1-i)}-1)!(m_i-l_1)!} \times\frac{(l_1+m_{2+\mathrm{U}(2-i)}-k-1)!\left(\frac{1}{\eta_i}-\frac{1}{\eta_{2+\mathrm{U}(2-i)}}\right)^{k-l_1-m_{2+\mathrm{U}(2-i)}}}{(m_{2+\mathrm{U}(2-i)}-1)!(l_1-k)!} \tag{A-8}$$

symbol. After some algebraic manipulations, (A-3) can be efficiently expressed as

$$f_{Z_2}(z) = \sum_{i=1}^{2}\sum_{k=1}^{m_i}\Xi_2(i,k,m_1,m_2,\eta_1,\eta_2)f_{Y_i}(z;k,\eta_i) \tag{A-4}$$

with Ξ_2 shown in (A-5) at the top of the page.

2) Step 2 ($L = 3$ Terms): For $L = 3$, the PDF of $Z_3 = Z_2 + Y_3$ can be evaluated using (A-4) as

$$f_{Z_3}(z) = \int_0^z f_{Z_2}(x)f_{Y_3}(z-x;m_3,\eta_3)dx. \tag{A-6}$$

By using [14, eq. (3.381.4)], and following similar steps as for the calculation of the PDF of Z_2, after some complicated, but also straightforward, manipulations, yields

$$f_{Z_3}(z)=\sum_{i=1}^{3}\sum_{k=1}^{m_i}\Xi_3(i,k,m_1,m_2,m_3,\eta_1,\eta_2,\eta_3,l_1)f_{Y_i}(z;k,\eta_i) \tag{A-7}$$

where Ξ_3 is shown in (A-8) at the top of the page.

3) Step 3 (L Terms): Following the same procedures as those in Steps 1 and 2 for the sum of L RVs, (6) can be extracted.[7]

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