

# On the Symbol Error Probability of General Order Rectangular QAM in Nakagami- $m$ Fading

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**Abstract**—Recently, Beaulieu, following an ingenious concept, presented a closed-form expression for a useful integral, which was used for the evaluation of the symbol error probability (SEP) of general order rectangular quadrature amplitude modulation (QAM) in slow Rayleigh fading. In this letter, these results are extended to Nakagami- $m$  fading channels, deriving a novel closed-form formula for the average over Nakagami- $m$  fading of the product of two Gaussian  $Q$ -functions, which can be efficiently used to study the impact of fading severity on the error performance of general rectangular QAM constellations.

**Index Terms**—Gaussian  $Q$ -function, quadrature amplitude modulation (QAM), Nakagami- $m$  fading channels, symbol error probability (SEP).

## I. INTRODUCTION

GENERAL order rectangular quadrature amplitude modulation (QAM) constellations can be easily generated from two independent pulse amplitude modulation (PAM) signals ( $M_I$ -PAM (in-phase) and  $M_Q$ -PAM (quadrature)) and can be also easily demodulated. Although, it is generally a sub-optimal modulation scheme, compared to other  $M$ -QAM constellations, in the sense that they do not maximally space the constellation points for a given energy, the average transmitted power required to achieve a given minimum distance is only slightly greater than the corresponding one required for the best  $M$ -QAM scheme. This is the reason that general order rectangular QAM modulation can be efficiently used in practical telecommunications systems [1].

Recently, Beaulieu in [2], presented a novel formula for the symbol error probability (SEP) of general rectangular QAM, in additive white Gaussian noise (AWGN). In the same work, he derived a closed-form expression for the integral

$$\Upsilon(a, b) = \int_0^{\infty} f_R(r) Q(ra) Q(rb) dr \quad (1)$$

where  $Q(x)$  is the Gaussian- $Q$  function [3],  $a, b$  are real constants and  $f_R(r)$  is the Rayleigh probability density function (pdf). This integral is involved in several wireless applications (see [2], [3] and the references therein) and was used in [2] to evaluate the average symbol error probability (ASEP) of general rectangular QAM modulations in slow Rayleigh fading.

In this paper, the approach of [2] is extended to Nakagami- $m$  fading channels. More specifically, a novel closed-form

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expression is presented for the average over Nakagami- $m$  fading of the product of two Gaussian  $Q$ -functions with dissimilar origins. This expression is used to evaluate in closed-form the ASEP of general order rectangular QAM, operating in Nakagami- $m$  fading channels and to present - via numerical examples- the impact of fading severity on the system's error performance. Finally, it is shown that the general results, presented in this letter, reduce to some specific cases previously published.

## II. THE AVERAGE OVER NAKAGAMI- $m$ FADING OF THE PRODUCT OF TWO GAUSSIAN $Q$ -FUNCTIONS

Consider the integral  $\Upsilon(a, b)$  in (1), but now assuming that  $R$  follows Nakagami- $m$  distribution with pdf given by [4]

$$f_R(r) = \frac{2m^m}{\Omega^m \Gamma(m)} r^{2m-1} \exp\left(-\frac{m}{\Omega} r^2\right) \quad (2)$$

where  $\Gamma(x)$  is the Gamma function [5, eq. (8.310/1)],  $\Omega = \mathcal{E}\langle R^2 \rangle$ ,  $m = \Omega^2 / \mathcal{E}\langle (R^2 - \Omega)^2 \rangle \geq 1/2$ , and  $\mathcal{E}\langle \cdot \rangle$  denotes expectation. The PDF in (2) includes the cases of Rayleigh ( $m = 1$ ) and one-sided Gaussian ( $m = 1/2$ ) distributions as special ones.

A closed-form solution for (1) for integer values of  $m$  and  $a = b$ , was given by Simon and Alouini in [3, ch. 5.1], while -as mentioned above- for arbitrary values of  $a, b$  and  $m = 1$ , Beaulieu presented in [2] a useful closed-form expression as

$$\Upsilon_{m=1}(a, b) = \frac{1}{4} - \frac{1}{2\pi h(a)} \tan^{-1}\left[\frac{a}{b} h(a)\right] - \frac{1}{2\pi h(b)} \tan^{-1}\left[\frac{b}{a} h(b)\right] \quad (3)$$

with  $h(x) = \sqrt{\frac{1+\sigma^2 x^2}{\sigma^2 x^2}}$  and  $\sigma^2 = \frac{\Omega}{2}$ . However, to the best of the author's knowledge, closed-form solution to (1) for the general case of Nakagami- $m$  fading channel with dissimilar values for  $a, b$ , does not exist in the literature.

In Appendix is proved that such a solution is

$$\Upsilon(a, b) = \frac{1}{4} - [g(a, b) + g(b, a)] \quad (4)$$

where

$$g(a, b) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{m-1} \frac{2^{k-1} m^k a \sqrt{\Omega} \Gamma(k + \frac{3}{2})}{(a^2 \Omega + 2m)^{k+\frac{1}{2}} k! (1+2k)} - \frac{1}{4\pi} \sum_{k=0}^{m-1} \frac{\Omega a b 2^{k+1} m^k {}_2F_1\left(1, 1+k; \frac{3}{2}; \frac{b^2 \Omega}{a^2 \Omega + 2m + b^2 \Omega}\right)}{(a^2 \Omega + 2m + b^2 \Omega)^{k+1}} \quad (5)$$

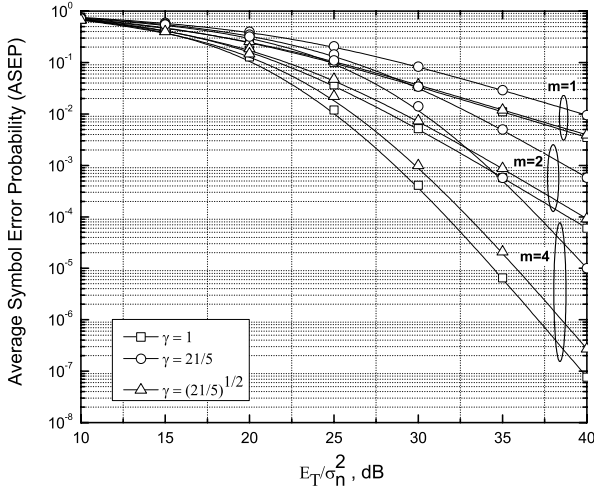


Fig. 1. ASEP for a 8x4-QAM in Nakagami- $m$  fading.

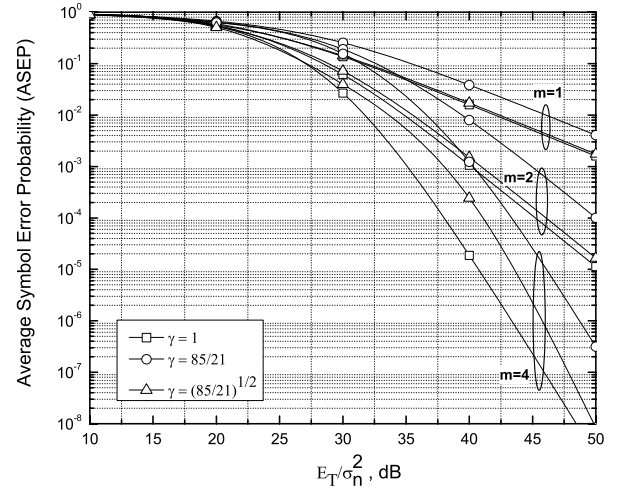


Fig. 2. ASEP for a 16x8-QAM in Nakagami- $m$  fading.

with  ${}_2F_1(x, y; z; w)$  being the Gauss Hypergeometric function [5] and  $m$  integer.

For the special case of  $m = 1$  (Rayleigh fading), using [5, eq. (9.121/14)] and after manipulations, (4) reduces to (3). Also, for  $a = b$  and integer  $m$ , using [5, eq. (9.121/14)], [6] and after laborious manipulations, (4) coincides to the already published result [3, (5.30)].

### III. THE IMPACT OF FADING SEVERITY ON THE ERROR PERFORMANCE OF GENERAL ORDER RECTANGULAR QAM

The SEP of general order rectangular QAM in AWGN is [2]

$$P_S(A_I, A_Q) = 2 \left(1 - \frac{1}{M_I}\right) Q(A_I) + 2 \left(1 - \frac{1}{M_Q}\right) Q(A_Q) - 4 \left(1 - \frac{1}{M_I}\right) \left(1 - \frac{1}{M_Q}\right) Q(A_I) Q(A_Q) \quad (6)$$

where  $M_I$ -PAM and  $M_Q$ -PAM are the in-phase and quadrature PAM signals,  $A_I = d_I/\sigma_n$  and  $A_Q = d_Q/\sigma_n$ , with  $d_I$ ,  $d_Q$  be the decision distances and  $\sigma_n^2$  the noise power.

The ASEP over Nakagami- $m$  fading channels can be written as

$$P = \int_0^\infty f_R(r) P_S(rA_I, rA_Q) dr \quad (7)$$

or

$$P = 2 \left(1 - \frac{1}{M_I}\right) \Psi(A_I) + 2 \left(1 - \frac{1}{M_Q}\right) \Psi(A_Q) - 4 \left(1 - \frac{1}{M_I}\right) \left(1 - \frac{1}{M_Q}\right) \Upsilon(A_I, A_Q) \quad (8)$$

with

$$\Psi(a) = \int_0^\infty Q(ar) f_R(r) dr \quad (9)$$

being the average over Nakagami- $m$  fading of  $Q(ax)$ . Fortunately, a closed-form solution to  $\Psi(a)$  for integer values of  $m$  was given in [3, eq. (5.17)] as

$$\Psi(a) = \frac{1}{2} - \frac{M(a)}{2} \sum_{k=0}^{m-1} \binom{2k}{k} \left(\frac{1 - M(a)^2}{4}\right)^k \quad (10)$$

where  $M(x) = \sqrt{\frac{x^2 \Omega}{1+x^2 \Omega}}$ . Substituting (4) and (10) in (8), the ASEP of general order rectangular QAM can be evaluated in closed-form. Next, using the above formulation, we give two examples for different rectangular constellations, to show the impact of fading severity on the system's error performance.

Let us consider a 8x4-QAM system, as in [2]. In Fig. 1 the ASEP is depicted as a function of the average symbol energy-to-noise power,  $\frac{E_T}{\sigma_n^2}$  (dB), where the signal-to-noise ratio (SNR) is

$$\frac{E_T}{\sigma_n^2} = 21 \Omega d_I^2 + 5 \Omega d_Q^2 = (21 + 5\gamma^2) \Omega A_I \quad (11)$$

with  $\gamma = \frac{d_Q}{d_I} = \frac{A_Q}{A_I}$ . It is evident that a decrease in the fading severity (greater  $m$ ) significantly affects the system's error performance. A transition from  $m = 1$  (Rayleigh) to  $m = 2$  leads to a performance improvement that ranges from 5 to 10 dB (depending on the  $\gamma$  and SNR), while from  $m = 1$  to  $m = 4$  this improvement may be greater than 10 dB. Moreover, it is interesting to observe that in low fading environments, the ASEP improvement is higher for the equal decision distance case ( $\gamma = 1$ ) compared to equal average energy ( $\gamma = \sqrt{21/5}$ ) in about 0.5 dB in Rayleigh fading to 2.5 dB for  $m = 4$ . Also, when the quadrature signal has 21/5 times the average energy of the in-phase signal ( $\gamma = 21/5$ ), the ASEP deteriorates substantially in lower fading conditions.

Similar observations can be obtained from Fig. 2, where a 16x8-QAM is studied. In this case, the SNR is

$$\frac{E_T}{\sigma_n^2} = 85 \Omega d_I^2 + 21 \Omega d_Q^2 = (85 + 21\gamma^2) \Omega A_I \quad (12)$$

It is interesting to note here that  $\gamma$  has a greater effect to the ASEP compared to 8x4-QAM, especially in higher values of  $m$ .

APPENDIX

Following the concept of [2, Appendix], the integral  $\Upsilon(a, b)$  in (1) can be splitted into two triple integrals, named  $\Upsilon_1(a, b)$  and  $\Upsilon_2(a, b)$ , as

$$\begin{aligned} \Upsilon(a, b) &= \int_0^\infty \int_0^{\frac{b}{a}x_1} \int_0^{\frac{x_2}{b}} f(x_1)f(x_2) f_R(r) dr dx_2 dx_1 \\ &+ \int_0^\infty \int_0^{\frac{a}{b}x_2} \int_0^{\frac{x_1}{a}} f(x_2)f(x_1) f_R(r) dr dx_1 dx_2 \\ &= \Upsilon_1(a, b) + \Upsilon_1(b, a) \end{aligned} \tag{13}$$

where  $f(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$ .

The inner integral in  $\Upsilon_1(a, b)$  is the Nakagami-*m* cumulative distribution function (cdf) at  $\frac{x_2}{b}$ , which can be solved using [5, eq. (3.381)], resulting in

$$\begin{aligned} \Upsilon_1(a, b) &= \frac{1}{2\pi \Gamma(m)} \\ &\int_0^\infty \int_0^{\frac{b}{a}x_1} f(x_1) f(x_2) \left[ \Gamma(m) - \Gamma\left(m, \frac{m x_2^2}{b^2 \Omega}\right) \right] dx_2 dx_1 \end{aligned} \tag{14}$$

with  $\Gamma(x, y)$  being the incomplete Gamma function [5, eq. (8.350.2)]. Substituting  $f(x_1)$  and  $f(x_2)$  in (14) one has

$$\begin{aligned} \Upsilon_1(a, b) &= \frac{1}{2\pi} \int_0^\infty \int_0^{\frac{b}{a}x_1} \exp\left(-\frac{x_1^2}{2} - \frac{x_2^2}{2}\right) dx_2 dx_1 - \frac{1}{2\pi \Gamma(m)} \\ &\int_0^\infty \int_0^{\frac{b}{a}x_1} \exp\left(-\frac{x_1^2}{2} - \frac{x_2^2}{2}\right) \Gamma\left(m, \frac{m x_2^2}{b^2 \Omega}\right) dx_2 dx_1. \end{aligned} \tag{15}$$

Now, after replacing the incomplete Gamma function with its finite series representation for integer values of *m* [5, (8.352/2)]

$$\begin{aligned} \Upsilon_1(a, b) &= \frac{1}{2\pi} \tan^{-1}\left(\frac{b}{a}\right) - \frac{(m-1)!}{2\pi \Gamma(m)} \\ &\int_0^\infty \int_0^{\frac{b}{a}x_1} \exp\left(-\frac{x_1^2 + x_2^2}{2} - \frac{m x_2^2}{b^2 \Omega}\right) \sum_{k=0}^{m-1} \frac{\left(\frac{m x_2^2}{b^2 \Omega}\right)^k}{k!} dx_2 dx_1 \end{aligned} \tag{16}$$

$$\begin{aligned} &= \frac{1}{2\pi} \tan^{-1}\left(\frac{b}{a}\right) - \sum_{k=0}^{m-1} \frac{m^k}{\left(\frac{b^2 \Omega + 2m}{2b^2 \Omega}\right)^{k+\frac{1}{2}} b^{2k} \Omega^k k! (2+4k)} \\ &[\sqrt{2\pi} \Gamma\left(k + \frac{3}{2}\right) - (2k+1) \\ &\int_0^\infty \exp\left(-\frac{x_1^2}{2}\right) \Gamma\left(k + \frac{1}{2}, \frac{b^2 \left(\frac{b^2 \Omega + 2m}{2b^2 \Omega}\right) x_1^2}{a^2}\right) dx_1]. \end{aligned} \tag{17}$$

The integral in (17) can be solved using [5, eq. (6.455)] resulting after manipulations in

$$\begin{aligned} \Upsilon_1(a, b) &= \frac{1}{2\pi} \tan^{-1}\left(\frac{b}{a}\right) \\ &- \frac{1}{\sqrt{\pi}} \sum_{k=0}^{m-1} \frac{2^{k-1} m^k b \sqrt{\Omega} \Gamma\left(k + \frac{3}{2}\right)}{(b^2 \Omega + 2m)^{k+\frac{1}{2}} k! (1+2k)} \\ &+ \frac{1}{4\pi} \sum_{k=0}^{m-1} \frac{\Omega b a 2^{k+1} m^k}{(b^2 \Omega + 2m + a^2 \Omega)^{k+1}} \\ &{}_2F_1\left(1, 1+k; \frac{3}{2}; \frac{a^2 \Omega}{b^2 \Omega + 2m + a^2 \Omega}\right). \end{aligned} \tag{18}$$

Now, using (13), (4) is proved.

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