On the Monotonicity of the Generalized Marcum and Nuttall Q-Functions

Vasilios M. Kapinas, *Member, IEEE*, Sotirios K. Mihos, *Student Member, IEEE*, and
George K. Karagiannidis, *Senior Member, IEEE*

Abstract—**Monotonicity criteria are established for the generalized Marcum Q-function, $Q_M(\alpha, \beta)$, the standard Nuttall Q-function, $Q_{M,N}(\alpha, \beta)$, and the normalized Nuttall Q-function, $\mathcal{Q}_{M,N}(\alpha, \beta)$, with respect to their real order indices M, N. Besides, closed-form expressions are derived for the computation of the standard and normalized Nuttall Q-functions for the case when M, N are odd multiples of 0.5 and $M \geq N$. By exploiting these results, novel upper and lower bounds for $Q_{M,N}(\alpha, \beta)$ and $\mathcal{Q}_{M,N}(\alpha, \beta)$ are proposed. Furthermore, specific tight upper and lower bounds for $Q_M(\alpha, \beta)$, previously reported in the literature, are extended for real values of M. The offered theoretical results can be efficiently applied in the study of digital communications over fading channels, in the information-theoretic analysis of multiple-input multiple-output systems and in the description of stochastic processes in probability theory, among others.**

Index Terms—**Closed-form expressions, generalized Marcum Q-function, lower and upper bounds, monotonicity, normalized Nuttall Q-function, standard Nuttall Q-function.**

I. INTRODUCTION

A. The Nuttall Q-Functions

AN extended version of the (standard) Marcum Q-function, $Q(\alpha, \beta) = \int_\beta^\infty x e^{-\frac{x^2+\alpha^2}{2}} I_0(\alpha x) dx$, where $\alpha, \beta \geq 0$, originally appeared in [1, Appendix, eq. (16)], defines the standard[1] Nuttall Q-function [2, eq. (86)], given by the integral representation

$$Q_{M,N}(\alpha, \beta) = \int_\beta^\infty x^M e^{-\frac{x^2+\alpha^2}{2}} I_N(\alpha x) dx \qquad (1)$$

where the order indices are generally reals with values $M \geq 0$ and $N > -1$, I_N is the Nth order modified Bessel function of the first kind [3, eq. (9.6.3)] and α, β are real parameters with $\alpha > 0$, $\beta \geq 0$. It is worth mentioning here, that the negative values of N, defined above, have not been of interest in any practical applications so far. However, the extension of the Nuttall

[1]We adopt the term "standard" for the Nuttall Q-function in order to avoid ambiguity with its normalized version to be introduced later.

Q-function to negative values of N has been introduced here in order to facilitate more effectively the relation of this function to the more common generalized Marcum Q-function, as will be shown in the sequel. An alternative version of $Q_{M,N}(\alpha, \beta)$ is the *normalized Nuttall Q-function*, $\mathcal{Q}_{M,N}(\alpha, \beta)$, which constitutes a normalization of the former with respect to the parameter α, defined simply by the relation

$$\mathcal{Q}_{M,N}(\alpha, \beta) \triangleq \frac{Q_{M,N}(\alpha, \beta)}{\alpha^N}. \qquad (2)$$

Typical applications involving the standard and normalized Nuttall Q-functions include: a) the error probability performance of noncoherent digital communication over Nakagami fading channels with interference [4], b) the outage probability of wireless communication systems where the Nakagami/Rician faded desired signals are subject to independent and identically distributed (i.i.d.) Rician/Nakagami faded interferers, respectively, under the assumptions of minimum interference and signal power constraints [4]–[7], c) the performance analysis and capacity statistics of uncoded multiple-input multiple-output (MIMO) systems operating over Rician fading channels [8]–[10], and d) the extraction of the required log-likelihood ratio for the decoding of differential phase-shift keying (DPSK) signals employing turbo or low-density parity-check (LDPC) codes [11].

Since both types of the Nuttall Q-function are not considered to be tabulated functions, their computation involved in the aforementioned applications was handled considering the two distinct cases of $M + N$ being either odd or even, in order to express them in terms of more common functions. The possibility of doing such when $M + N$ is odd was suggested in [2], requiring particular combination of the two recursive relations [2, eqs. (87) and (88)]. However, the explicit solution was derived only in [4, eq. (13)] entirely in terms of the Marcum Q-function and a finite weighted sum of modified Bessel functions of the first kind. Having all the above in mind, along with the fact that the calculation of $Q(\alpha, \beta)$ itself requires numerical integration, the issue of the efficient computation of (1) and (2) still remains open.

B. The Generalized Marcum Q-Function

The generalized Marcum Q-function [12] of positive real order M, is defined by the integral [13, eq. (1)]

$$Q_M(\alpha, \beta) \triangleq \frac{1}{\alpha^{M-1}} \int_\beta^\infty x^M e^{-\frac{x^2+\alpha^2}{2}} I_{M-1}(\alpha x) dx \qquad (3)$$

where α, β are nonnegative real parameters[2]. For $M = 1$, it reduces to the popular standard (or first-order) Marcum Q-function, $Q_1(\alpha, \beta)$ (or $Q(\alpha, \beta)$), while for general M it is related to the normalized Nuttall Q-function according to [14, eq. (4.105)]

$$Q_M(\alpha, \beta) = \mathcal{Q}_{M,M-1}(\alpha, \beta), \quad \alpha > 0. \qquad (4)$$

An identical function to the generalized Marcum Q is the probability of detection[3] [1, eq. (49)], which has a long history in radar communications and particularly in the study of target detection by pulsed radar with single or multiple observations [1], [16]–[18]. Additionally, $Q_M(\alpha, \beta)$ is strongly associated with a) the error probability performance of noncoherent and differentially coherent modulations over generalized fading channels [14], [19]–[23], b) the signal energy detection of a primary user over a multipath channel [24], [25], and finally (c) the information-theoretic study of MIMO systems [26]. Aside from these applications, the generalized Marcum Q-function presents a variety of interesting probabilistic interpretations. Most indicatively, for integer M, it is the complementary cumulative distribution function (CCDF) of a noncentral chi-square (χ^2) random variable with $2M$ degrees of freedom (DOF) [27, eq. (2.45)]. This relationship was extended in [28] to work for the case of odd DOF as well, through a generalization of the noncentral χ^2 CCDF. Similar relations can be found in the literature involving the generalized Rician [29, eq. (2.1–145)], the generalized Rayleigh [30, pp. 1] (for $\alpha = 0$) and the bivariate Rayleigh [31, Appendix A],[32] (for $M = 1$) CCDF's. Finally, in a recent work [33], a new association has been derived between the generalized Marcum Q-function and a probabilistic comparison of two independent Poisson random variables.

More than thirty algorithms have been proposed in the literature for the numerical computation of the standard and generalized Marcum Q-functions, among them power series expansions [34]–[36], approximations and asymptotic expressions [37]–[40], and Neumann series expansions [41]–[43]. However, the above representations may not always provide sufficient information about the relative position of the approximated value with respect to the exact one, which in some applications is highly desired. In [44], the generalized Marcum Q-function of integer order M has been expressed as a single integral with finite limits, which is computationally more desirable relatively to other methods suggested previously. Nevertheless, the integral cannot be computed analytically and appropriate numerical integration techniques have to be applied, thereby introducing an approximation error in its computation. In [45], an exact representation for $Q_M(\alpha, \beta)$, when M is an odd multiple of 0.5, was given as a finite sum of tabulated functions, assuming that $\beta^2 > \alpha^2 + 2M$. This result was recently enhanced in [46] to a single expression that remains accurate over all ranges of the parameters α, β, while in [47] the same expression was bounded by particular utilization of novel Gaussian Q-function inequalities. Finally, in [48], an equivalent expression to [46, eq. (11)]

was derived, adopting a completely different (analytical) approach from the latter.

Close inspection of the issues mentioned above, render the existence of upper and lower bounds a matter of essential importance in the computation of (3). Several types of bounds for the standard [49]–[51] and generalized [37], [46], [48], [52]–[54] Marcum Q-functions have been suggested so far. However, all the aforementioned works consider just integer values of M, which is generally true when this parameter represents the number of independent samples of a square-law detector output. Nevertheless, in many applications, this requirement does not hold. According to [14, Sec. 4.4.2], it would be desirable to obtain alternative representations for $Q_M(\alpha, \beta)$ regardless of whether M is integer or not. For instance, in [20], the fading parameter of the Nakagami-m distribution is restricted to integer values in the lack of a closed-form expression for the generalized Marcum Q-function of real order. Additionally, in [25], [55]–[57], the order M of the generalized Marcum Q-function, involved in the energy detection in various radiometer and cognitive radio applications, is expressed as the product of the integration time and the receiver bandwidth, thus implying that in general M is a non-integer quantity. Furthermore, a probabilistic interpretation of $Q_{M-\mu}(\alpha, \beta)$, where $M \in \mathbb{N}$[4] and $\mu = 0.5$, is given in [58], where it is related to main probabilistic characteristics of $2(M - \mu)$ random variables, while in [59]–[61], noncentral χ^2 random variables with fractional DOF are studied. Finally, in [62], the integrand of (3) has been proved to be a probability density function (pdf) for $\alpha \geq 0$ and $M > 0$, a result that also has been utilized in [48].

C. Contribution

As described in Section I-A, a closed-form expression for the computation of the standard and normalized Nuttall Q-functions is available in the literature only for the case of odd $M+N$, with the additional restriction of integers M, N. In Section II-A, we derive a novel closed-form expression for the computation of $Q_{M,N}(\alpha, \beta)$ and $\mathcal{Q}_{M,N}(\alpha, \beta)$ when M, N are odd multiples of 0.5 and $M \geq N$, being valid for all ranges of the parameters α, β.

Besides, in Section II-B, we proceed with the establishment of appropriate monotonicity criteria, revealing the behavior of both functions with the sum $M + N$. Specifically, we demonstrate that the standard Nuttall Q-function is strictly increasing with respect to $M + N$ when $M \geq N + 1$, under the constraints of $\alpha \geq 1$ and $\beta > 0$. For the normalized Nuttall Q-function, a similar monotonicity statement is proved without the necessity of reducing the range of α.

An alternative approach, sufficient enough to facilitate the problem of evaluating the Nuttall Q-functions, is the derivation of tight bounds. Nevertheless, to the best of the authors' knowledge, such bounds have not been reported in the literature so far. Section II-B is completed with the exploitation of the previous results in order to derive novel upper and lower bounds for $Q_{M,N}(\alpha, \beta)$ and $\mathcal{Q}_{M,N}(\alpha, \beta)$ when $M \geq N + 1$ and $\beta > 0$, with the extra requirement of $\alpha \geq 1$ for the former.

---

[2]For $\alpha = 0$ the right hand side of (3) can be easily shown to satisfy the limiting value of [14, eq. (4.71)], reproduced in (30).

[3]For M incoherently integrated signals, the two functions are simply related by $Q_M(\alpha, \beta) = P_M\left(\frac{\alpha^2}{2M}, \frac{\beta^2}{2}\right)$, as induced by [15, eq. (7)].

[4]Throughout the manuscript, we adopt $\mathbb{N}$ and $\mathbb{N}_0$ notations for the representation of the positive and the nonnegative integer set, respectively. In the same way, $\mathbb{R}^+$ includes the positive and $\mathbb{R}_0^+$ the nonnegative reals.

Additionally, in Section I-B, the need for computing the generalized Marcum Q-function, $Q_M(\alpha, \beta)$, of real order M was highlighted, since it is a case of frequent occurrence in various applications. However, a thorough literature search for studies concerning arbitrary values of M, revealed only [46] for the closed-form computation of $Q_M(\alpha, \beta)$ of half-odd integer order, and the accepted paper [63], where bounds for $Q_M(\alpha, \beta)$ were introduced for the case when M is not necessarily an integer. These considerations motivated us to generalize the scope of M in [46, eq. (16)]

$$Q_{M-0.5}(\alpha, \beta) < Q_M(\alpha, \beta) < Q_{M+0.5}(\alpha, \beta), \quad M \in \mathbb{N} \tag{5}$$

as described in Section III, by providing a monotonicity formalization for the generalized Marcum Q-function, namely that $Q_M(\alpha, \beta)$ is strictly increasing with respect to its order $M > 0$ for $\alpha \geq 0$ and $\beta > 0$. This interesting statement was also recently presented in [62], using a different approach. As a consequence, novel upper and lower bounds for $Q_M(\alpha, \beta)$ of positive real order are derived. We finalize the paper with some concluding remarks, given in Section IV.

II. MONOTONICITY OF THE NUTTALL Q-FUNCTIONS

A. Novel Closed-Form Representations

So far, closed-form expression for either type of the Nuttall Q-function is not available in the literature. In this section, we derive such a representation for the case when M, N are odd multiples of 0.5 and $M \geq N$, through the theorem and corollary established below. Before proceeding further with the corresponding proofs, some definitions of essential functions and notations used, would be very convenient.

Hereafter, Γ, γ and $\Gamma(\cdot, \cdot)$ will denote the Euler gamma [3, eq. (6.1.1)], the lower incomplete gamma [3, eq. (6.5.2)] and the upper incomplete gamma [3, eq. (6.5.3)] functions, respectively, defined by the integrals

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \ \gamma(z, x) = \int_0^x t^{z-1} e^{-t} dt$$
$$\Gamma(z, x) = \Gamma(z) - \gamma(z, x), \quad z \in \mathbb{R}^+, x \in \mathbb{R}.$$

Notations $n!$, $(m)_n$ and $\binom{m}{n}$ imply the factorial [3, eq. (6.1.6)], the rising factorial (Pochhammer's symbol) [3, eq. (6.1.22)] and the binomial coefficient [3, eq. (24.1.1 C)], respectively, defined by $n! = \prod_{k=1}^n k$ for $n \in \mathbb{N}$; $= 1$ for $n = 0$, $(m)_n = \frac{(m+n-1)!}{(m-1)!}$ for $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ for $m, n \in \mathbb{N}_0$, $m \geq n$. Finally, $\mathrm{sgn}(z) = z/|z|$ for $z \neq 0$; $= 0$ for $z = 0$, stands for the signum function.

Theorem 1 (Closed-Form for the Standard Nuttall Q): The standard Nuttall Q-function, $Q_{M,N}(\alpha, \beta)$, when $m = M + 0.5 \in \mathbb{N}$, $n = N + 0.5 \in \mathbb{N}$ and $M \geq N$, can be evaluated for $\alpha > 0$, $\beta \geq 0$ by the following closed-form expression:

$$Q_{M,N}(\alpha, \beta) = \frac{(-1)^n (2\alpha)^{-n+\frac{1}{2}}}{\sqrt{\pi}}$$
$$\times \sum_{k=0}^{n-1} \frac{(n-k)_{n-1}(2\alpha)^k}{k!} \mathcal{I}_{m,n}^k(\alpha, \beta)$$

where the term $\mathcal{I}_{m,n}^k(\alpha, \beta)$ is given by

$$\mathcal{I}_{m,n}^k(\alpha, \beta) = (-1)^{k+1} \sum_{l=0}^{m-n+k} \binom{m-n+k}{l} 2^{\frac{l-1}{2}} \alpha^{m-n+k-l}$$
$$\times \left[(-1)^{m-n-l-1} \Gamma\left(\frac{l+1}{2}, \frac{(\beta+\alpha)^2}{2}\right) \right.$$
$$- (\mathrm{sgn}(\beta-\alpha))^{l+1} \gamma\left(\frac{l+1}{2}, \frac{(\beta-\alpha)^2}{2}\right)$$
$$\left. + \Gamma\left(\frac{l+1}{2}\right) \right]. \tag{6}$$

Proof: Given that $n = N + 0.5 \in \mathbb{N}$, the modified Bessel function of the first kind, I_N, can be expressed by the finite sum [64, eq. (8.467)], which after some manipulations can be written as

$$I_N(z) = \frac{(-1)^n (2z)^{-n+\frac{1}{2}}}{\sqrt{\pi} e^z} \sum_{k=0}^{n-1} \frac{(n-k)_{n-1}(2z)^k}{k!}$$
$$\times \left(1 - (-1)^k e^{2z}\right), \ n = N + \frac{1}{2} \in \mathbb{N}, z \in \mathbb{R}. \tag{7}$$

Therefore, using (1) and (7), the standard Nuttall Q-function satisfies

$$Q_{M,N}(\alpha, \beta) = \frac{(-1)^n (2\alpha)^{-n+\frac{1}{2}}}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{(n-k)_{n-1}(2\alpha)^k}{k!}$$
$$\times \left[\int_\beta^\infty x^{m-n+k} e^{-\frac{(x+\alpha)^2}{2}} dx \right.$$
$$\left. - (-1)^k \int_\beta^\infty x^{m-n+k} e^{-\frac{(x-\alpha)^2}{2}} dx \right]. \tag{8}$$

The calculation of the integral difference in (8) can be effectively facilitated by the following definition

$$\mathcal{I}_L^k(\alpha, \beta) = \int_\beta^\infty x^L e^{-\frac{(x+\alpha)^2}{2}} dx - (-1)^k \int_\beta^\infty x^L e^{-\frac{(x-\alpha)^2}{2}} dx \tag{9}$$

where $L = m-n+k$. Since we examine the case when $M \geq N$ or equivalently $m \geq n$, it follows that in the above expression the exponent L is a nonnegative integer. Therefore, using [65, eq. (1.3.3.18)], (9) obtains the form
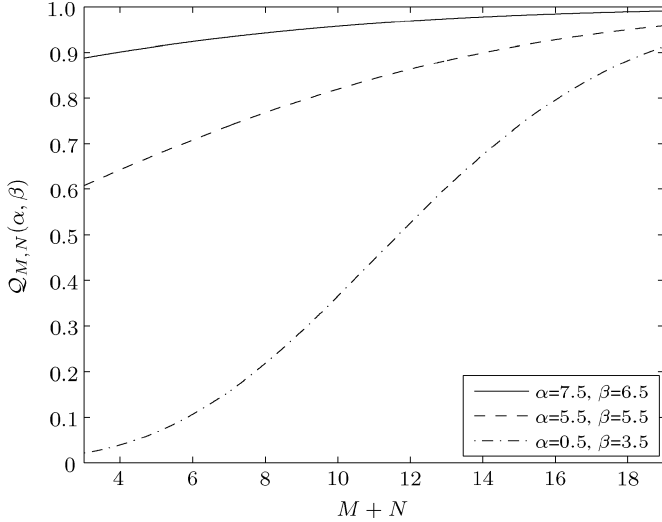
$$\mathcal{I}_L^k(\alpha, \beta) = \sum_{l=0}^L \binom{L}{l} \alpha^{L-l} \left[(-1)^{L-l} \int_{\beta+\alpha}^\infty x^l e^{-\frac{x^2}{2}} dx \right.$$
$$\left. - (-1)^k \int_{\beta-\alpha}^\infty x^l e^{-\frac{x^2}{2}} dx \right]. \tag{10}$$

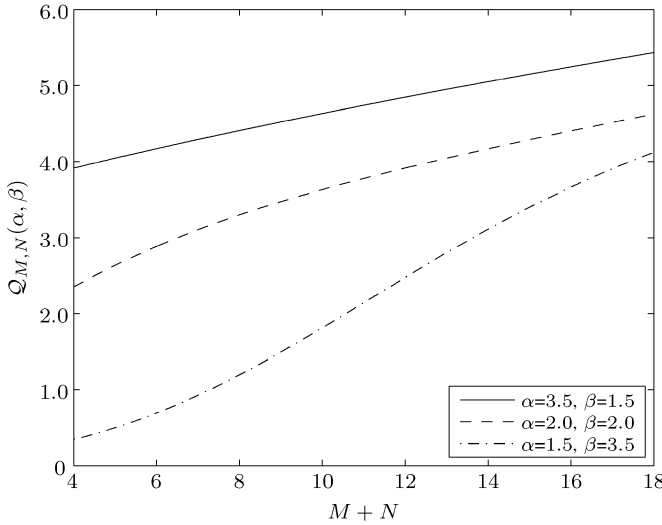The two integrals involved in (10) can be considered as special cases of the more general one

$$\mathcal{I}_b^l = \int_b^\infty x^l e^{-\frac{x^2}{2}} dx, \quad b \in \mathbb{R}$$

which for the case of nonnegative values of b can be calculated from [64, eq. (3.381.3)] as

$$\mathcal{I}_b^l = 2^{\frac{l-1}{2}} \Gamma\left(\frac{l+1}{2}, \frac{b^2}{2}\right), \quad b \geq 0 \tag{11}$$

(a) $\mathcal{Q}_{M,N}(\alpha, \beta)$ versus $M + N$ for $M - N = 1$.

(b) $\mathcal{Q}_{M,N}(\alpha, \beta)$ versus $M + N$ for $M - N = 2$.

Fig. 1. Monotonicity of $\mathcal{Q}_{M,N}(\alpha, \beta)$ with respect to the sum $M + N$ for several real values of α, β.

while for negative values of b, [64, eqs. (3.381.1), (3.381.4)] can be combined to yield

$$\mathcal{I}_b^l = 2^{\frac{l-1}{2}} \left[\Gamma \left(\frac{l+1}{2} \right) + (-1)^l \gamma \left(\frac{l+1}{2}, \frac{b^2}{2} \right) \right], \quad b < 0. \tag{12}$$

Therefore, a single expression for the integral $\mathcal{I}_b^l$ for any real value of b can be derived, by merging (11) and (12) with the help of [64, eq. (8.356.3)], in order to satisfy

$$\mathcal{I}_b^l = 2^{\frac{l-1}{2}} \left[\Gamma \left(\frac{l+1}{2} \right) - [\text{sgn}(b)]^{l+1} \gamma \left(\frac{l+1}{2}, \frac{b^2}{2} \right) \right].$$

Thus, (9) is equivalent to

$$\mathcal{I}_L^k(\alpha, \beta) = \sum_{l=0}^{L} \binom{L}{l} \alpha^{L-l} \left[(-1)^{L-l} \mathcal{I}_{\beta+\alpha}^l - (-1)^k \mathcal{I}_{\beta-\alpha}^l \right]$$

$$= (-1)^{k+1} \sum_{l=0}^{L} \binom{L}{l} 2^{\frac{l-1}{2}} \alpha^{L-l} \left[\Gamma \left(\frac{l+1}{2} \right) \right.$$
$$+ (-1)^{L-l-k-1} \Gamma \left(\frac{l+1}{2}, \frac{(\beta+\alpha)^2}{2} \right)$$
$$\left. - [\text{sgn}(\beta - \alpha)]^{l+1} \gamma \left(\frac{l+1}{2}, \frac{(\beta-\alpha)^2}{2} \right) \right]$$

which, after the substitution $L = m - n + k$, yields (6), thus completing the proof. ∎

Corollary 1 (Closed-Form for the Normalized Nuttall Q): The normalized Nuttall Q-function, $\mathcal{Q}_{M,N}(\alpha, \beta)$, when $m = M + 0.5 \in \mathbb{N}$, $n = N + 0.5 \in \mathbb{N}$ and $M \geq N$, can be evaluated for $\alpha > 0$, $\beta \geq 0$ by the following closed-form expression:

$$\mathcal{Q}_{M,N}(\alpha, \beta) = \frac{(-1)^n 2^{-n+\frac{1}{2}}}{\sqrt{\pi} \alpha^{2n-1}} \sum_{k=0}^{n-1} \frac{(n-k)_{n-1} (2\alpha)^k}{k!} \mathcal{I}_{m,n}^k(\alpha, \beta)$$

where the term $\mathcal{I}_{m,n}^k(\alpha, \beta)$ is given by (6).

Proof: The proof follows immediately from (2) and Theorem 1. ∎

B. Lower and Upper Bounds

In this section, novel lower and upper bounds for the normalized and standard Nuttall Q-functions are proposed.

Lemma 1: The function $\mathcal{G}_s(r, x)$, defined by

$$\mathcal{G}_s(r, x) \triangleq \frac{\Gamma(r+s, x)}{\Gamma(r)}, \quad r, x \in \mathbb{R}^+ \tag{13}$$

is strictly increasing with respect to r for all $s \in \mathbb{R}_0^+$.

Proof: By multiplying both the numerator and denominator of (13) by the upper incomplete gamma function, $\Gamma(r, x)$, we obtain

$$\mathcal{G}_s(r, x) = \frac{\Gamma(r+s, x)}{\Gamma(r, x)} \mathcal{G}_0(r, x)$$

where from (13) one can observe that the term $\mathcal{G}_0(r, x)$ is the complement of the regularized lower incomplete gamma function $P(r, x)$ with respect to unity, defined in [3, eq. (6.5.1)] by $P(r, x) = \frac{\gamma(r, x)}{\Gamma(r)}$ for all $r > 0$ and $x \in \mathbb{R}$. Fortunately, $P(r, x)$ for $r, x > 0$ is equal to the cumulative distribution function (CDF) of the standard gamma distribution $\text{Gamma}(r, 1)$, which is strictly decreasing with respect to the shape parameter r. Additionally, this important result has also been proved analytically in [66, eq. (59)], thus implying that $\mathcal{G}_0(r, x)$ is strictly increasing with respect to $r > 0$ for all $x > 0$. Furthermore, in [67], it has been demonstrated that the function

$$\mathcal{R}(p, q, x) = \left[\frac{\Gamma(p, x)}{\Gamma(q, x)} \right]^{\frac{1}{p-q}}, \quad p > q > 0, x > 0 \tag{14}$$

is increasing with respect to q. By substituting $p = r + s$ and $q = r$ into (14), we realize that the ratio $\Gamma(r+s, x)/\Gamma(r, x)$ is increasing with respect to r for $s > 0$, while it remains constant

for the trivial case of $s = 0$. Therefore, it increases with $r > 0$ for all $x > 0$, $s \geq 0$, and the proof is complete. $\square$

The outcome of Lemma 1 will be utilized for the establishment of the next theorem, concerning the monotonicity property of the normalized Nuttall Q-function.

Theorem 2 (Monotonicity of the Normalized Nuttall Q): The normalized Nuttall Q-function, $\mathcal{Q}_{M,N}(\alpha, \beta)$, where $M > 0$, $N > -1$ and $\alpha, \beta > 0$, is strictly increasing with respect to the sum $M + N$, under the requirement of constant difference $M - N \geq 1$.

Proof: Combining (1), (2) and using the series representation of the modified Bessel function of the first kind in terms of the gamma function [64, eq. (8.445)], we obtain

$$\mathcal{Q}_{M,N}(\alpha, \beta) = e^{-\frac{\alpha^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{k! \Gamma(k + N + 1) 2^{2k+N}} \\ \times \int_{\beta}^{\infty} x^{2k+M+N} e^{-\frac{x^2}{2}} dx \quad (15)$$

where we have interchanged the order of integration and summation, since all integrand quantities of the normalized Nuttall Q-function are Riemann integrable on $[\beta, \infty)$. Additionally, the integral in (15) is the case of (11), thus yielding

$$\int_{\beta}^{\infty} x^{2k+M+N} e^{-\frac{x^2}{2}} dx = 2^{k + \frac{M+N-1}{2}} \Gamma\left(k + \frac{M+N+1}{2}, \frac{\beta^2}{2}\right).$$

Therefore, (15) reads

$$\mathcal{Q}_{M,N}(\alpha, \beta) = e^{-\frac{\alpha^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{2^{k + \frac{N-M+1}{2}} k!} \frac{\Gamma\left(k + \frac{M+N+1}{2}, \frac{\beta^2}{2}\right)}{\Gamma(k + N + 1)}. \quad (16)$$

Introducing the variables $v = M + N$ and $c = M - N$ and taking the partial derivative of both sides of (16) with respect to v, we can easily obtain

$$\frac{\partial}{\partial v} \mathcal{Q}_{\frac{v+c}{2}, \frac{v-c}{2}}(\alpha, \beta) = e^{-\frac{\alpha^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{2^{k + \frac{3-c}{2}} k!} \\ \times \frac{\partial}{\partial u(v)} \mathcal{G}_{\frac{c-1}{2}}\left(u(v), \frac{\beta^2}{2}\right) \quad (17)$$

where the function $u(v) = k + 1 + \frac{v-c}{2}$ has been employed for notational convenience. We note here that, applying the Weierstrass M-test [68], the series in (17) can be proved to converge uniformly, thus enabling one to interchange the order of differentiation and summation. Hence, recalling Lemma 1 and the requirement of $\alpha > 0$, that follows from the definition of the normalized Nuttall Q-function, the proof is complete. $\square$

In Fig. 1(a) and (b), the normalized Nuttall Q-function has been plotted versus the sum $M + N$ for several values of α, β, considering $M - N = 1$ and $M - N = 2$, respectively. However, we note here that Theorem 2 implies non-integer differences $M - N$ as well.

For the interpretation of the next proposition, we define the pair of half-integer rounding operators $\lfloor x \rceil_{0.5}$ and $\lceil x \rceil_{0.5}$ that map a real x to its nearest left and right half-odd integer, respectively, according to the relations[5]

$$\lfloor x \rceil_{0.5} = \lfloor x - 0.5 \rfloor + 0.5$$
$$\lceil x \rceil_{0.5} = \lceil x + 0.5 \rceil - 0.5 \quad (18)$$

where $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the integer floor and ceiling functions. Additionally, we recall that if $\delta_x \in [0, 1)$ is the fractional part of x, then $\lfloor x \rfloor = x - \delta_x$.

Corollary 2 (Bounds on the Normalized Nuttall Q): The following inequalities can serve as lower and upper bounds on the normalized Nuttall Q-function, $\mathcal{Q}_{M,N}(\alpha, \beta)$, where $\alpha, \beta > 0$ and $M, N > 0.5$, for the case when $M \geq N + 1$ and $\delta_M = \delta_N$ (i.e., $M - N \in \mathbb{N}$)

$$\mathcal{Q}_{M,N}(\alpha, \beta) \geq \mathcal{Q}_{\lfloor M \rceil_{0.5}, \lfloor N \rceil_{0.5}}(\alpha, \beta)$$
$$\mathcal{Q}_{M,N}(\alpha, \beta) \leq \mathcal{Q}_{\lceil M \rceil_{0.5}, \lceil N \rceil_{0.5}}(\alpha, \beta) \quad (19)$$

with the equality above being valid only for the case of half-odd integer values of M, N.

Proof: The proof follows immediately from Theorem 2. $\square$

For the calculation of the bounds in (19), the quantities $\mathcal{Q}_{\lfloor M \rceil_{0.5}, \lfloor N \rceil_{0.5}}(\alpha, \beta)$ and $\mathcal{Q}_{\lceil M \rceil_{0.5}, \lceil N \rceil_{0.5}}(\alpha, \beta)$ can be evaluated exactly by utilizing the results of Corollary 1. Moreover, for the case of $M, N \in \mathbb{N}$, the proposed bounds obtain the simplified form

$$\mathcal{Q}_{M,N}(\alpha, \beta) > \mathcal{Q}_{M-0.5, N-0.5}(\alpha, \beta)$$
$$\mathcal{Q}_{M,N}(\alpha, \beta) < \mathcal{Q}_{M+0.5, N+0.5}(\alpha, \beta). \quad (20)$$

In Fig. 2(a) and (b), the normalized Nuttall Q-function along with its lower and upper bounds are depicted versus β for several values of α and M, respectively, while the parameter N is restricted according to the relation $N = M - c$ with $c \in \mathbb{N}$ taking values $c = 2$ in Fig. 2(a) and $c = 1, 2, 3$ in Fig. 2(b). It is evident, that the bounds proposed in (19) are very tight, especially the upper one for $\delta_M (= \delta_N) < 0.5$ and the lower one for $\delta_M (= \delta_N) > 0.5$, the latter being the case illustrated in Fig. 2(b).

In order to obtain lower and upper bounds for the standard Nuttall Q-function, a similar procedure can be carried out. The next theorem will be particularly useful for the fulfillment of such a derivation.

Theorem 3 (Monotonicity of the Standard Nuttall Q): The standard Nuttall Q-function, $Q_{M,N}(\alpha, \beta)$, where $M > 0$, $N > -1$ and $\alpha \geq 1$, $\beta > 0$, is strictly increasing with respect to the sum $M + N$, under the requirement of constant difference $M - N \geq 1$.
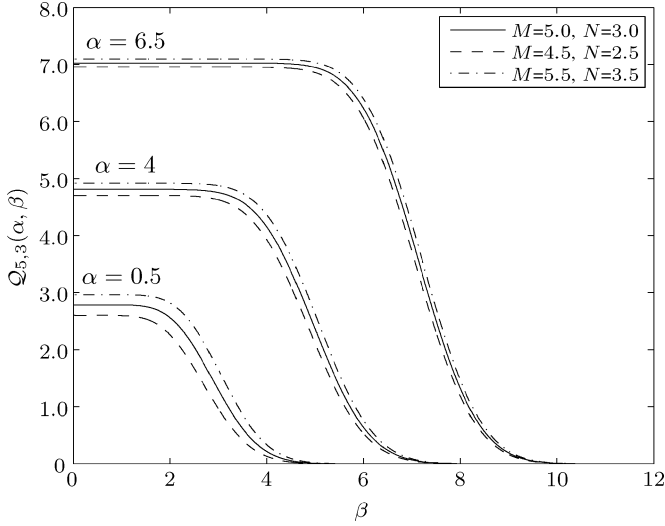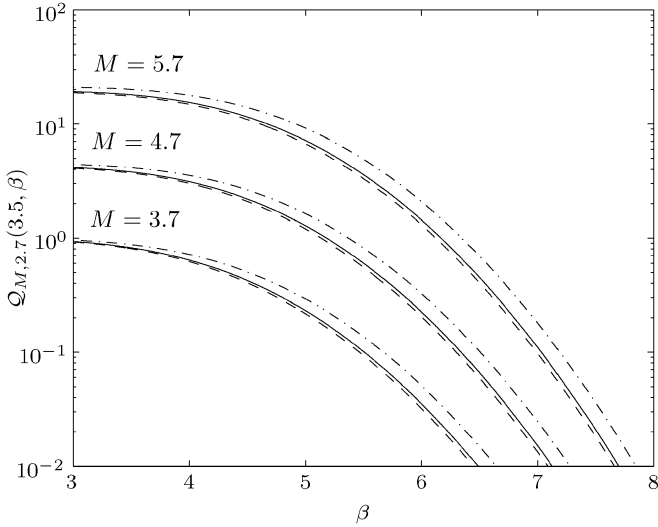
Proof: In Theorem 2, it has been proved that

$$\frac{\partial}{\partial v} \mathcal{Q}_{\frac{v+c}{2}, \frac{v-c}{2}}(\alpha, \beta) > 0 \quad (21)$$

where we have substituted $v = M + N$ and $c = M - N$. From (2) and (21), and after using the quotient rule for partial differentiation, we obtain

$$\frac{\partial}{\partial v} Q_{\frac{v+c}{2}, \frac{v-c}{2}}(\alpha, \beta) > \frac{\ln \alpha}{2} Q_{\frac{v+c}{2}, \frac{v-c}{2}}(\alpha, \beta).$$

[5]The defining equations of (18) can be easily verified to be valid in any arbitrary segment $[n, n + 1)$, where $n \in \mathbb{Z}$.

(a) $\mathcal{Q}_{5,3}(\alpha,\beta)$ versus β for several values of α.

(b) $\mathcal{Q}_{M,2.7}(3.5,\beta)$ versus β for several values of M.

Fig. 2. Bounds of $\mathcal{Q}_{M,N}(\alpha,\beta)$ for $M-N \in \mathbb{N}$ and several values of α, β.

Since the Nuttall Q-function is strictly positive, then for $\alpha \geq 1$ it follows that

$$\frac{\partial}{\partial v} Q_{\frac{v+c}{2},\frac{v-c}{2}}(\alpha,\beta) > 0$$

and the proof is complete. $\square$

Corollary 3 (Bounds On the Standard Nuttall Q): The following inequalities can serve as lower and upper bounds on the standard Nuttall Q-function, $Q_{M,N}(\alpha,\beta)$, where $\alpha \geq 1, \beta > 0$ and $M, N > 0.5$, for the case when $M \geq N+1$ and $\delta_M = \delta_N$ (i.e., $M - N \in \mathbb{N}$)

$$Q_{M,N}(\alpha,\beta) \geq Q_{\lfloor M \rfloor_{0.5},\lfloor N \rfloor_{0.5}}(\alpha,\beta)$$
$$Q_{M,N}(\alpha,\beta) \leq Q_{\lceil M \rceil_{0.5},\lceil N \rceil_{0.5}}(\alpha,\beta) \qquad (22)$$

with the equality above being valid only for the case of half-odd integer values of M, N.

Proof: The proof follows immediately from Theorem 3. $\square$

Similarly to the case of the normalized Nuttall Q, in the calculation of the bounds from (22), the quantities $Q_{\lfloor M \rfloor_{0.5},\lfloor N \rfloor_{0.5}}(\alpha,\beta)$ and $Q_{\lceil M \rceil_{0.5},\lceil N \rceil_{0.5}}(\alpha,\beta)$ can be evaluated exactly from Theorem 1. Finally, for $M, N \in \mathbb{N}$, the standard Nuttall Q-function can be simply bounded by

$$Q_{M,N}(\alpha,\beta) > Q_{M-0.5,N-0.5}(\alpha,\beta)$$
$$Q_{M,N}(\alpha,\beta) < Q_{M+0.5,N+0.5}(\alpha,\beta) \qquad (23)$$

which constitutes the counterpart of (20) for the standard Nuttall Q-function.

III. Monotonicity and Bounds for the Generalized Marcum Q-Function

Recently, Li and Kam in [46, eq. (11)], following a geometric approach, presented a novel closed-form formula for the evaluation of $Q_M(\alpha,\beta)$, for the case when M is an odd multiple of 0.5 and $\alpha > 0, \beta \geq 0$, given by

$$Q_M(\alpha,\beta) = \frac{1}{2}\mathrm{erfc}\left(\frac{\beta+\alpha}{\sqrt{2}}\right) + \frac{1}{2}\mathrm{erfc}\left(\frac{\beta-\alpha}{\sqrt{2}}\right)$$
$$+ \frac{1}{\alpha\sqrt{2\pi}} \sum_{k=0}^{M-1.5} \frac{\beta^{2k}}{2^k} \sum_{q=0}^{k} \frac{(-1)^q(2q)!}{(k-q)!q!}$$
$$\times \sum_{i=0}^{2q} \frac{1}{(\alpha\beta)^{2q-i}i!}\left[(-1)^i e^{-\frac{(\beta-\alpha)^2}{2}} - e^{-\frac{(\beta+\alpha)^2}{2}}\right]$$
$$(24)$$

where $\mathrm{erfc}(z) = (2/\sqrt{\pi})\int_z^\infty e^{-t^2}dt$ is the complementary error function [3, eq. (7.1.2)]. This representation involves only elementary functions and is convenient for evaluation both numerically and analytically. For the trivial case when $\alpha = 0$, exact values of the generalized Marcum Q-function can be obtained from [46, eq. (12)]

$$Q_M(0,\beta) = \mathrm{erfc}\left(\frac{\beta}{\sqrt{2}}\right) + \frac{e^{-\frac{\beta^2}{2}}}{\sqrt{2\pi}} \sum_{k=0}^{M-1.5} \frac{\beta^{2k+1}}{2^{k-1}}$$
$$\times \sum_{q=0}^{k} \frac{(-1)^q}{(k-q)!q!(2q+1)}. \qquad (25)$$

Following an algebraic approach, an alternative more compact closed-form expression, equivalent to (24), can be derived, considering the next steps. Particularly, in [58, eq. (10)] it has been proved that the generalized Marcum Q-function of order $m-\mu$, with m positive integer and $0 \leq \mu < 1$ can be written in terms of the generalized Marcum Q-function of order $1-\mu$ as

$$Q_{m-\mu}(\alpha,\beta) = e^{-\frac{\alpha^2+\beta^2}{2}} \sum_{n=1}^{m-1}\left(\frac{\beta}{\alpha}\right)^{n-\mu} I_{n-\mu}(\alpha\beta)$$
$$+ Q_{1-\mu}(\alpha,\beta), \quad \alpha \neq 0.$$

By substituting $\mu = 0.5$ in the above equation and noting that for this case the modified Bessel function of the first kind can be replaced by (7), we obtain

$$Q_{m-0.5}(\alpha,\beta) = \alpha\sqrt{\frac{2}{\pi}}e^{-\frac{(\alpha+\beta)^2}{2}} \sum_{n=1}^{m-1}(-2\alpha^2)^{-n}$$
$$\times \sum_{k=0}^{n-1} \frac{(n-k)_{n-1}}{k!}(2\alpha\beta)^k\left[1-(-1)^k e^{2\alpha\beta}\right]$$
$$+ Q_{0.5}(\alpha,\beta), \quad m \in \mathbb{N} \qquad (26)$$

where once again $(m)_n$ denotes the Pochhammer's symbol and the term $Q_{0.5}(\alpha, \beta)$ can be derived from the definition of the generalized Marcum Q-function in (3), by using [3, eq. (10.2.14)] as follows

$$Q_{0.5}(\alpha, \beta) = \sqrt{\frac{2}{\pi}} \int_\beta^\infty e^{-\frac{x^2+\alpha^2}{2}} \cosh(ax) dx.$$

The above integral can be computed in closed-form as

$$Q_{0.5}(\alpha, \beta) = \frac{1}{2}\text{erfc}\left(\frac{\beta+\alpha}{\sqrt{2}}\right) + \frac{1}{2}\text{erfc}\left(\frac{\beta-\alpha}{\sqrt{2}}\right)$$
$$= Q(\beta+\alpha) + Q(\beta-\alpha) \quad (27)$$

where Q denotes the Gaussian Q-function (or Gaussian probability integral) [3, eq. (26.2.3)], defined by $Q(z) = (1/\sqrt{2\pi}) \int_z^\infty e^{-t^2/2} dt$. Using (26) and (27), the generalized Marcum Q-function of half-odd integer order can be computed for all $\alpha > 0, \beta \geq 0$ from the expression

$$Q_M(\alpha, \beta) = \alpha\sqrt{\frac{2}{\pi}} e^{-\frac{(\alpha+\beta)^2}{2}} \sum_{n=1}^{M-0.5} (-2\alpha^2)^{-n}$$
$$\times \sum_{k=0}^{n-1} \frac{(n-k)_{n-1}}{k!} (2\alpha\beta)^k \left[1-(-1)^k e^{2\alpha\beta}\right]$$
$$+ Q(\beta+\alpha) + Q(\beta-\alpha), \quad M+0.5 \in \mathbb{N}. \quad (28)$$

We note here that a similar result to (28) has been recently reported in the literature [48, eq. (16)]. In order to examine the special case when $\alpha = 0$, we first notice that from (4) and (16) an alternative expression—equivalent to [3, eq. (26.4.25)]—for the generalized Marcum Q-function can be derived, written as

$$Q_M(\alpha, \beta) = e^{-\frac{\alpha^2}{2}} \sum_{k=0}^\infty \frac{\alpha^{2k}}{2^k k!} \frac{\Gamma\left(k+M, \frac{\beta^2}{2}\right)}{\Gamma(k+M)}, \quad \alpha > 0, \beta \geq 0 \quad (29)$$

which for integer M falls into the series expansion [35, eq. (4)]. Since $Q_M(\alpha, \beta)$ is a continuous function of α for all $\beta \geq 0$ and $M > 0$, the above equation can be extended to be asymptotically valid for the case when $\alpha = 0$ as well, with the corresponding limiting value given by

$$Q_M(0, \beta) = \frac{\Gamma\left(M, \frac{\beta^2}{2}\right)}{\Gamma(M)}. \quad (30)$$

This last result also appears in [14, eq. (4.71)], where $Q_M(0, \beta)$ has been derived directly from (3) by applying the small argument form of the modified Bessel function.

It has been proved in [46] that (24), (25) along with (5) can define tight upper and lower bounds for the generalized Marcum Q-function of integer order. It seems apparent, that in order to derive bounds for $Q_M(\alpha, \beta)$ of real order M, a strict inequality, involving the whole range of M, has to be established. Such a generalization concept can be formalized through the following theorem.

Theorem 4 (Monotonicity of the Generalized Marcum Q): The generalized Marcum Q-function, $Q_M(\alpha, \beta)$, is strictly increasing with respect to its real order $M > 0$ for all $\alpha \geq 0, \beta > 0$.

Proof: Concerning the case when $\alpha = 0$, we notice that (30) can be rewritten as

$$Q_M(0, \beta) = 1 - P\left(M, \frac{\beta^2}{2}\right).$$

However, in [66, eq. (59)] the regularized lower incomplete gamma function $P(r, x)$ has been proved to decrease monotonically with respect to $r > 0$ for all $x > 0$. Additionally, for $\alpha > 0$, (4) implies that the normalized Nuttall Q-function with $N = M - 1$ falls into the generalized Marcum Q-function of order M. Nevertheless, according to Theorem 2, $\mathcal{Q}_{M,M-1}(\alpha, \beta)$ is strictly increasing with respect to $2M - 1$ for $M > 0$, and the proof is complete. $\square$

The result of Theorem 4 has also recently demonstrated by Sun and Baricz in [62], where two totally different proofs were given. The first one combines the series form of the generalized Marcum Q-function presented in (29) and (30) together with the fact that the regularized upper incomplete gamma function $Q(r, x) = 1 - P(r, x)$ is strictly increasing with respect to $r > 0$ for each $x > 0$, originally stated by Tricomi in [66]. A slightly different analytical proof to this can also be found in [60, Th. 1]. The second proof exploits the interesting relationship between the generalized Marcum Q-function and the reliability function (or CCDF) R of a χ^2 random variable with $2M$ DOF and noncentrality parameter α, namely the fact that if $\beta \sim \chi^2_{2M,\alpha}$ then $R(\beta) = Q_M(\sqrt{\alpha}, \sqrt{\beta})$. The interested reader is referred to [62, Th. 3.1] for more information.

Recalling the relation between the normalized Nuttall and the generalized Marcum Q-functions, that is (4), Fig. 1(a) verifies graphically the results of Theorem 4, since it actually depicts $Q_M(\alpha, \beta)$ versus the term $2M - 1$.

Corollary 4 (Bounds On the Generalized Marcum Q): The following inequalities can serve as lower and upper bounds on the generalized Marcum Q-function $Q_M(\alpha, \beta)$ of real order $M > 0.5$ for all $\alpha \geq 0, \beta > 0$.

$$Q_{\lfloor M \rfloor_{0.5}}(\alpha, \beta) < Q_M(\alpha, \beta) \leq Q_{\lceil M \rceil_{0.5}}(\alpha, \beta) \quad (31)$$

with the equality above being valid only for the case of half-odd integer values of M.

Proof: The proof follows immediately from Theorem 4. $\square$

In Corollary 4, the quantities $Q_{\lfloor M \rfloor_{0.5}}(\alpha, \beta)$ and $Q_{\lceil M \rceil_{0.5}}(\alpha, \beta)$ can be evaluated exactly either from (24), (25) or (27), (30), while for $M \in \mathbb{N}$ (31) reduces to

$$Q_{M-0.5}(\alpha, \beta) < Q_M(\alpha, \beta) < Q_{M+0.5}(\alpha, \beta)$$

which comes as a complement to the inequalities of (20) and (23). This last result was originally demonstrated in [46, eq. (16)], where the authors following a geometric approach proposed tight lower and upper bounds for the generalized Marcum Q-function of integer order M, which have been proved to outperform other existing ones. This can be easily verified from Fig. 3(a), where $Q_4(\alpha, \beta)$ has been plotted versus β for several values of α. Therefore, for the case of real M, one can expect even further enhancement in the strictness of either the lower
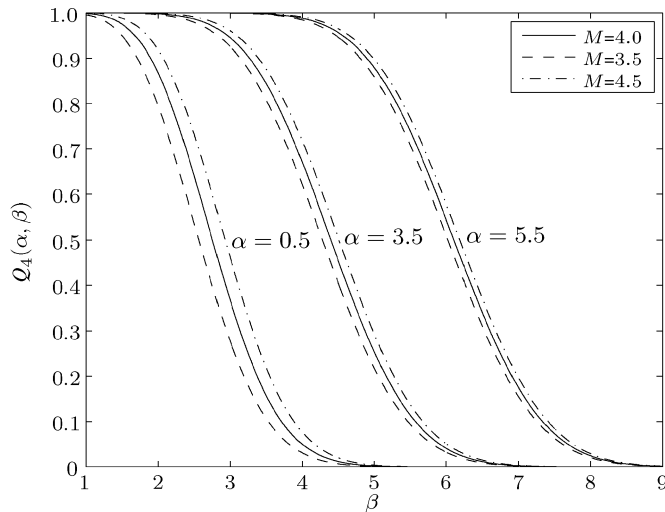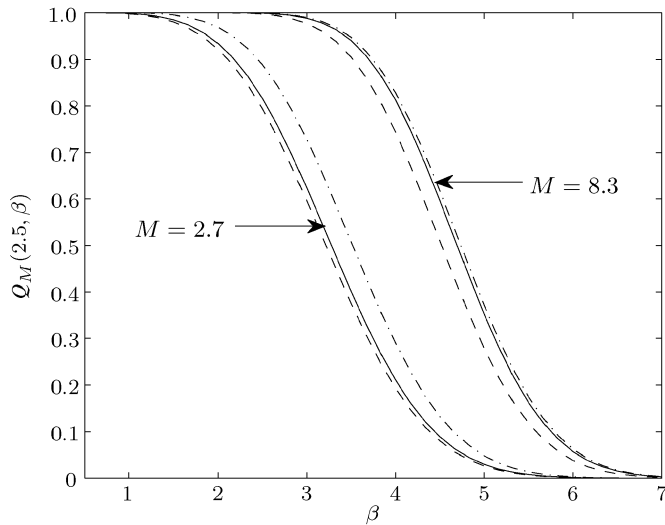
(a) $Q_4(\alpha, \beta)$ versus β for several values of α.

(b) $Q_M(2.5, \beta)$ versus β for several values of M.

Fig. 3. Bounds of $Q_M(\alpha, \beta)$ for several real values of α, β and M. (In Fig. 3(a) bounds proposed by Li and Kam [46].)

bound (for $\delta_M > 0.5$) or the upper one (for $\delta_M < 0.5$). This is clearly depicted in Fig. 3(b), where the curves $Q_{2.5}(2.5, \beta)$ and $Q_{8.5}(2.5, \beta)$ constitute very tight lower and upper bounds of $Q_{2.7}(2.5, \beta)$ and $Q_{8.3}(2.5, \beta)$, respectively, for all range of β.

IV. CONCLUSION

Applicable monotonicity criteria were established for the normalized and standard Nuttall and the generalized Marcum Q-functions. Specifically, it was proved that the two Nuttall Q-functions are strictly increasing with respect to the real sum $M + N$ for the case when $M \geq N + 1$, while the generalized Marcum Q-function increases monotonically with respect to its real order M. Additionally, novel closed-form expressions for both types of the Nuttall Q-function were given for the case when M, N are odd multiples of 0.5 and $M \geq N$. Regarding the generalized Marcum Q-function of half-odd integer order, an alternative more compact closed-form expression, equivalent

to the already existing one, was derived. By exploiting these results, novel lower and upper bounds were proposed for the Nuttall Q-functions when $M \geq N + 1$, while the recently proposed bounds for the generalized Marcum Q-function of integer M, were appropriately utilized in order to extend their validity over real values of M.

REFERENCES

[1] J. I. Marcum, "A statistical theory of target detection by pulsed radar," *IRE Trans. Inf. Theory*, vol. IT-6, no. 2, pp. 59–267, Apr. 1960.
[2] H. Nuttall, Some Integrals Involving the Q-Function Naval Underwater Systems Center, New London, CT, Tech. Rep. 4297, 1972.
[3] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th ed.   New York: Dover, 1972.
[4] M. K. Simon, "The Nuttall Q function—Its relation to the Marcum Q function and its application in digital communication performance evaluation," *IEEE Trans. Commun.*, vol. 50, pp. 1712–1715, Nov. 2002.
[5] H.-C. Yang and M.-S. Alouini, "Closed-form formulas for the outage probability of wireless communication systems with a minimum signal power constraint," *IEEE Trans. Veh. Technol.*, vol. 51, pp. 1689–1698, Nov. 2002.
[6] L. Yang and M.-S. Alouini, "On the average outage rate and average outage duration of wireless communication systems with multiple cochannel interferers," *IEEE Trans. Wireless Commun.*, vol. 3, no. 4, pp. 1142–1153, Jul. 2004.
[7] J. M. Romero-Jerez and A. J. Goldsmith, "Antenna array processing in fading and interference: An interference-cancellation vs. diversity comparative performance," in *Proc. IEEE Int. Conf. Commun. (ICC'07)*, Glasgow, Scotland, Jun. 24–28, 2007, pp. 1108–1113.
[8] M. Kang and M.-S. Alouini, "Largest eigenvalue of complex Wishart matrices and performance analysis of MIMO MRC systems," *IEEE J. Sel. Areas Commun.*, vol. 21, no. 3, pp. 418–426, Apr. 2003.
[9] S. Jin, M. R. Mckay, X. Gao, and I. B. Collings, "MIMO multichannel beamforming: SER and outage using new eigenvalue distributions of complex noncentral Wishart matrices," *IEEE Trans. Commun.*, vol. 56, pp. 424–434, Mar. 2008.
[10] A. Maaref and S. Aissa, "Joint and marginal eigenvalue distributions of (non)central complex Wishart matrices and PDF-based approach for characterizing the capacity statistics of MIMO Ricean and Rayleigh fading channels," *IEEE Trans. Wireless Commun.*, vol. 6, no. 10, pp. 3607–3619, Oct. 2007.
[11] K.-P. Ho and H.-C. Wang, "Soft-decoding of low-density parity check codes for DPSK signals," in *Proc. Opt. Fiber Commun./Nat. Fiber Optic Eng. Conf. (OFC/NFOEC'07)*, Anaheim, CA, Mar. 25–29, 2007, pp. 1–3.
[12] J. I. Marcum, Table of Q Functions Rand Corp., Santa Monica, CA, Res. Memo. RM-339, 1950.
[13] A. H. Nuttall, "Some integrals involving the Q_M function," *IEEE Trans. Inf. Theory*, vol. 21, no. 1, pp. 95–96, Jan. 1975.
[14] M. K. Simon and M.-S. Alouini, *Digital Communication over Fading Channels*, 2nd ed.   Hoboken, NJ: Wiley-Interscience, 2005.
[15] D. A. Shnidman, "The calculation of the probability of detection and the generalized Marcum Q-function," *IEEE Trans. Inf. Theory*, vol. 35, no. 2, pp. 389–400, Mar. 1989.
[16] P. Swerling, "Probability of detection for fluctuating targets," *IRE Trans. Inf. Theory*, vol. IT-6, no. 2, pp. 269–308, Apr. 1960.
[17] C. W. Helstrom, *Statistical Theory of Signal Detection*, 2nd ed.   New York: Pergamon, 1968.
[18] H. L. V. Trees, *Detection, Estimation, and Modulation Theory, Part I: Detection, Estimation and Linear Modulation Theory*.   New York: Wiley-Interscience, 2001.
[19] M. K. Simon and M.-S. Alouini, "A unified approach to the probability of error for noncoherent and differentially coherent modulations over generalized fading channels," *IEEE Trans. Commun.*, vol. 46, pp. 1625–1638, Dec. 1998.

[20] M. K. Simon and M.-S. Alouini, "A unified performance analysis of digital communication with dual selective combining diversity over correlated Rayleigh and Nakagami-m fading channels," *IEEE Trans. Commun.*, vol. 47, pp. 33–43, Jan. 1999.

[21] M. K. Simon and M.-S. Alouini, "Some new results for integrals involving the generalized Marcum Q function and their application to performance evaluation over fading channels," *IEEE Trans. Wireless Commun.*, vol. 2, no. 4, pp. 611–615, Jul. 2003.

[22] S. Gaur and A. Annamalai, "Some integrals involving the $Q_m(a\sqrt{x}, b\sqrt{x})$ with application to error probability analysis of diversity receivers," *IEEE Trans. Veh. Technol.*, vol. 52, pp. 1568–1575, Nov. 2003.

[23] A. Annamalai, C. Tellambura, and V. K. Bhargava, "A general method for calculating error probabilities over fading channels," *IEEE Trans. Commun.*, vol. 53, pp. 841–852, May 2005.

[24] F. F. Digham, M.-S. Alouini, and M. K. Simon, "On the energy detection of unknown signals over fading channels," *IEEE Trans. Commun.*, vol. 55, pp. 21–24, Jan. 2007.

[25] A. Pandharipande and J.-P. M. G. Linnartz, "Performance analysis of primary user detection in a multiple antenna cognitive radio," in *Proc. IEEE Int. Conf. Commun. (ICC'07)*, Glasgow, Scotland, Jun. 24–28, 2007, pp. 6482–6486.

[26] Y. Zhu, P. Y. Kam, and Y. Xin, "On the mutual information distribution of MIMO Rician fading channels," *IEEE Trans. Commun.*, to be published.

[27] M. K. Simon, *Probability Distributions Involving Gaussian Random Variables: A Handbook for Engineers and Scientists*. New York: Springer Science-Business Media, 2006.

[28] D. A. Shnidman, "Two noncentral chi-square generalizations," *IEEE Trans. Inf. Theory*, vol. 42, no. 1, pp. 283–285, Jan. 1996.

[29] J. G. Proakis, *Digital Communications*, 4th ed. New York: McGraw-Hill, 2000.

[30] J. Omura and T. Kailath, Some Useful Probability Distributions Stanford Univ., Syst. Theory Lab., Stanford, CA, Tech. Rep. 7050–6, 1965.

[31] M. Schwartz, W. R. Bennett, and S. Stein, *Communication Systems and Techniques*. New York: McGraw-Hill, 1966.

[32] M. K. Simon and M.-S. Alouini, "A simple single integral representation of the bivariate Rayleigh distribution," *IEEE Commun. Lett.*, vol. 2, pp. 128–130, May 1998.

[33] G. V. Weinberg, "Poisson representation and Monte Carlo estimation of generalized Marcum Q-function," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 42, pp. 1520–1531, Oct. 2006.

[34] G. H. Robertson, "Computation of the noncentral chi-square distribution," *Bell Syst. Tech. J.*, vol. 48, pp. 201–207, Jan. 1969.

[35] G. M. Dillard, "Recursive computation of the generalized Q function," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-9, pp. 614–615, Jul. 1973.

[36] D. A. Shnidman, "Efficient evaluation of probabilities of detection and the generalized Q-function," *IEEE Trans. Inf. Theory*, vol. IT-22, no. 6, pp. 746–751, Nov. 1976.

[37] S. S. Rappaport, "Computing approximations for the generalized Q function and its complement," *IEEE Trans. Inf. Theory*, vol. IT-17, no. 4, pp. 497–498, Jul. 1971.

[38] C. W. Helstrom, "Computing the generalized Marcum Q-function," *IEEE Trans. Inf. Theory*, vol. 38, no. 4, pp. 1422–1428, Jul. 1992.

[39] G. V. Weinberg and L. Panton, Numerical Estimation of Marcum's Q-Function using Monte Carlo Approximation Schemes Defence Science and Technology Organisation, Edinburgh, Australia, Res. Rep. DSTO-RR-0311, 2006.

[40] N. Ding and H. Zhang, "A flexible method to approximate Marcum Q-function based on geometric way of thinking," in *Proc. Int. Symp. Commun., Contr. Signal Process. (ISCCSP'08)*, St. Julians, Malta, Mar. 12–14, 2008, pp. 1351–1356.

[41] D. E. Johansen, New Techniques for Machine Computation of the Q-Functions, Truncated Normal Deviates, and Matrix Eigenvalues Appl. Res. Lab., Sylvania Electron. Syst., Waltham, MA, Tech. Rep. AFCRL-556, 1961.

[42] S. Parl, "A new method of calculating the generalized Q function," *IEEE Trans. Inf. Theory*, vol. IT-26, no. 1, pp. 121–124, Jan. 1980.

[43] P. E. Cantrell, "On the calculation of the generalized Q-function via Parl's method," *IEEE Trans. Inf. Theory*, vol. IT-32, no. 6, pp. 817–824, Nov. 1986.

[44] M. K. Simon, "A new twist on the Marcum Q-function and its application," *IEEE Commun. Lett.*, vol. 2, pp. 39–41, Feb. 1998.

[45] A. H. M. Ross, "Algorithm for calculating the noncentral chi-square distribution," *IEEE Trans. Inf. Theory*, vol. 45, no. 4, pp. 1327–1333, May 1999.

[46] R. Li and P. Y. Kam, "Computing and bounding the generalized Marcum Q-function via a geometric approach," in *Proc. IEEE Int. Symp. Inf. Theory (ISIT'06)*, Seattle, WA, Jul. 9–14, 2006, pp. 1090–1094.

[47] G. T. F. de Abreu, "Jensen-Cotes upper and lower bounds on the Gaussian Q-function and related functions," *IEEE Trans. Commun.*, submitted for publication.

[48] Y. Sun and S. Zhou, "Tight bounds of the generalized Marcum Q-function based on log-concavity," in *Proc. IEEE Global Telecommun. Conf. (GLOBECOM'08)*, New Orleans, LA, Nov. 4, 2008, pp. 1–5.

[49] G. E. Corazza and G. Ferrari, "New bounds for the Marcum Q-function," *IEEE Trans. Inf. Theory*, vol. 48, no. 11, pp. 3003–3008, Nov. 2002.

[50] P. Y. Kam and R. Li, "Computing and bounding the first-order Marcum Q-function: A geometric approach," *IEEE Trans. Commun.*, vol. 56, pp. 1101–1110, Jul. 2008.

[51] X. Zhao, D. Gong, and Y. Li, "Tight geometric bound for Marcum Q-function," *Electron. Lett.*, vol. 44, no. 5, pp. 340–341, Feb. 2008.

[52] R. Li and P. Y. Kam, "Generic exponential bounds on the generalized Marcum Q-function via the geometric approach," in *Proc. IEEE Global Telecommun. Conf. (GLOBECOM'07)*, Wash., DC, Nov. 26–30, 2007, pp. 1754–1758.

[53] M. K. Simon and M.-S. Alouini, "Exponential-type bounds on the generalized Marcum Q-function with application to error probability analysis over fading channels," *IEEE Trans. Commun.*, vol. 48, pp. 359–366, Mar. 2000.

[54] A. Annamalai and C. Tellambura, "Cauchy-Schwarz bound on the generalized Marcum Q-function with applications," *Wireless Commun. Mob. Comput.*, vol. 1, no. 2, pp. 243–253, Mar. 2001.

[55] R. F. Mills and G. E. Prescott, "A comparison of various radiometer detection models," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 32, pp. 467–473, Jan. 1996.

[56] C. W. Helstrom, "Approximate inversion of Marcum's Q-function," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 34, pp. 317–319, Jan. 1998.

[57] S. P. Herath and N. Rajatheva, "Analysis of equal gain combining in energy detection for cognitive radio over Nakagami channels," in *Proc. IEEE Global Telecommun. Conf. (GLOBECOM'08)*, New Orleans, LA, Nov. 4, 2008, pp. 1–5.

[58] G. Di Blasio and A. Neri, "Modified Q-functions and their use in detection analysis," *IEEE Trans. Inf. Theory*, vol. 37, no. 4, pp. 1123–1142, Jul. 1991.

[59] P. B. Patnaik, "The noncentral χ^2- and F-distributions and their applications," *Biometrika*, vol. 36, no. 1/2, pp. 202–232, Jun. 1949.

[60] B. K. Ghosh, "Some monotonicity theorems for χ^2, F and t distributions with applications," *J. Roy. Statist. Soc. B*, vol. 35, no. 3, pp. 480–492, 1973.

[61] N. M. Temme, "Asymptotic and numerical aspects of the noncentral chi-square distribution," *Comput. Math. Appl.*, vol. 25, no. 5, pp. 55–63, 1993.

[62] Y. Sun and Á. Baricz, "Inequalities for the generalized Marcum Q-function," *Appl. Math. Comput.*, vol. 203, no. 1, pp. 134–141, Sep. 2008.

[63] Á. Baricz and Y. Sun, "New bounds for the generalized Marcum Q-function," *IEEE Trans. Inf. Theory*, vol. 55, no. 7, pp. 3091–3100, Jul. 2009.

[64] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. San Diego, CA: Academic, 2000.

[65] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series*, 3rd ed. New York: Gordon and Breach Science, 1992, vol. 1, Elementary Functions.

[66] F. G. Tricomi, "Sulla funzione gamma incompleta," *Ann. Mat. Pura Appl.*, vol. 31, no. 1, pp. 263–279, Dec. 1950.

[67] F. Qi, "Monotonicity results and inequalities for the gamma and incomplete gamma functions," *Math. Inequal. Appl.*, vol. 5, no. 1, pp. 61–68, 2002.

[68] E. C. Titchmarsh, *The Theory of Functions*, 2nd ed. London, U.K.: Oxford Univ. Press, 1939.

Vasilios M. Kapinas (S'07–M'09) was born in Thessaloniki, Greece, in May 1976. He received the diploma degree in electrical and computer engineering from Aristotle University of Thessaloniki, Greece, in 2000.

Since 2005, he has been working toward the Ph.D. degree in telecommunications engineering. His current research interests include wireless communication theory and digital communications over fading channels, giving special focus to space-time block coding techniques.

Sotirios K. Mihos (S'XX) was born in Thessaloniki, Greece, in April 1984.

He is an undergraduate student at the Aristotle University of Thessaloniki, Greece, where he is working toward the diploma degree in electrical and computer engineering. His research interests span a wide range of subject areas including computer science, electronics and automatic control, with a special focus on their relationship to pure mathematics.

George K. Karagiannidis (M'97–SM'04) was born in Pithagorion, Samos Island, Greece. He received the University and Ph.D. degrees in electrical engineering from the University of Patras, Patras, Greece, in 1987 and 1999, respectively.

From 2000 to 2004, he was a Senior Researcher with the Institute for Space Applications and Remote Sensing, National Observatory of Athens, Greece. In June 2004, he joined Aristotle University of Thessaloniki, Thessaloniki, Greece, where he is currently an Assistant Professor with the Electrical and Computer Engineering Department. His current research interests include wireless communication theory, digital communications over fading channels, cooperative diversity systems, cognitive radio, satellite communications, and wireless optical communications. He is the author or coauthor of more than 80 technical papers published in scientific journals and presented at international conferences. He is also a coauthor of two chapters in books and a coauthor of the Greek edition of a book on mobile communications.

Dr. Karagiannidis has been a member of Technical Program Committees for several IEEE conferences. He is a member of the editorial boards of the IEEE TRANSACTIONS ON COMMUNICATIONS and the IEEE COMMUNICATIONS LETTERS. He is a corecipient of the Best Paper Award of the Wireless Communications Symposium (WCS) in IEEE International Conference on Communications (ICC' 07), Glasgow, U.K., June 2007. He is a full member of Sigma Xi. He serves on the editorial board of the EURASIP JOURNAL ON WIRELESS COMMUNICATIONS AND NETWORKING.