θ-QAM: A Parametric Quadrature Amplitude Modulation Family and its Performance in AWGN and Fading Channels

Koralia N. Pappi, *Student Member, IEEE*, Athanasios S. Lioumpas, *Student Member, IEEE,* and George K. Karagiannidis, *Senior Member, IEEE*

***Abstract*—We study a parametric quadrature amplitude modulation (QAM) family, called θ-QAM, which includes other known constellations, such as square QAM (SQAM) and triangular QAM (TQAM), as special cases. The versatile structure of the θ-QAM signal constellation, which occurs from the varying angle between the signal points, results in achieving the minimum symbol error rate (SER) or bit error rate (BER), under an average power constraint. The theoretical study aims at providing insight into the trade-off between error performance and complexity of this parametric modulation scheme. Exact analytical expressions are obtained for the SER in additive white Gaussian Noise (AWGN) and Nakagami-m fading channels, while highly accurate BER approximations are presented. Finally, we find the optimum angles, in a minimal SER or BER sense, for a specific signal-to-noise ratio (SNR) and modulation order, M. This serves as an indicator for the appropriate constellation with respect to channel conditions and SER or BER requirements. The presented theoretical analysis is validated via extensive computer simulations.**

***Index Terms*—Quadrature Amplitude Modulation (QAM), θ-QAM, square-QAM, triangular-QAM, Gray code penalty.**

I. Introduction

THE utilization of higher-order modulation formats is one of the most common methods for increasing the bit rate in communications systems without increasing the required bandwidth. In the early 1960's, particular attention had been turned to the family of suppressed carrier two-dimensional signal constellations, such as the combined amplitude and phase modulation (AM-PM) [1]- [3] and the quadrature amplitude modulation (QAM) [4], [5]. Some specific constellations attracted special interest, mainly due to the low complexity demodulation methods required, such as the square-QAM (SQAM) and have found many applications in practical systems. One of the most important applications of QAM are the adaptive modulation techniques [6]-[8], which aim at the real-time balancing of the link budget through adaptive variation of the transmitted power level, symbol transmission rate, constellation size, coding rate/scheme, or any combination of these parameters. Adaptive modulation techniques can also be combined with other famous communication techniques, such as antenna diversity [9].

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The authors are with the Department of Electrical and Computer Engineering, Aristotle University of Thessaloniki, 54124 Thessaloniki, Greece (e-mail: {kpappi, alioumpa, geokarag}@auth.gr).

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In the past, considerable efforts have been concentrated on finding the QAM constellations with the minimum error probability under an average power constraint. Towards this concept, Foschini et al. [10], by employing asymptotic approximations for the high signal-to-noise ratio (SNR) region, used a gradient-search procedure, which was initiated from several randomly chosen N-point arrays and converged in each case to a locally optimum constellation. Moreover, they rigorously proved that the optimum constellations formed a lattice of (almost) equilateral triangles, which offer a 0.5 dB improvement in SNR over the conventional QAM signaling formats, considering additive white Gaussian noise (AWGN) channels. Later, in [11], the authors studied via simulations the symbol error rate (SER) of various 16-ary and 64-ary QAM constellations, such as the circular-QAM (CQAM), the triangular-QAM (TQAM), also called hexagonal packing and the rectangular-QAM (RQAM), concluding that TQAM results in the best SER for average power-limited systems and they proposed a bit-to-symbol mapping through a heuristic technique. Recently, Conti et al. proposed an analytical framework based on the log-concavity property of the error probability and constructed a class of local bounds for the error probability for a wide family of multidimensional modulation formats in the presence of Gaussian disturbances and fading [12] . In [13], [14], the author studied the performance of TQAM, suggesting a detection method and a novel bit-to-symbol mapping. However, TQAM has an important disadvantage: the bit-to-symbol mapping cannot be optimal as in SQAM, resulting in a Gray code penalty, which deteriorates the bit error rate (BER). The performance of QAM schemes has been extensively studied also in numerous works (for example see [15]-[18]).

In this paper, we study a parametric QAM family, called θ-QAM, which includes the SQAM and TQAM as special cases. For a specific value of the SNR, the maximum transmission efficiency is achieved by a different member of this family. The SER performance in AWGN is expressed by exact analytical formulas, and the BER by accurate approximations. Furthermore, we evaluate the SER and BER in Nakagami-m fading, deriving exact closed-form expressions for the former and an approximation for the latter. Ascertaining on these expressions, we calculate the optimum angles, which minimize the SER or the BER for specific signal-to-noise ratio (SNR) values and modulation order, M. The theoretical study aims at providing insight into the trade-off between complexity and performance over AWGN and fading channels, of this parametric modulation scheme, which includes the practical cases of TQAM and SQAM as special cases.

The rest of the paper is as follows. In Section II, we present the θ-QAM family and derive the exact SER in AWGN. In Section III, the optimum θ-QAM constellations are studied, while in Section IV we present an efficient approximation for the BER. In Section V, we evaluate the SER and BER performance in Nakagami-m fading. Finally, concluding remarks are presented in Section VI.

II. The θ-QAM Family

Let us consider an M-ary SQAM constelaltion, where binary data are mapped into M two-tuples (x_m, y_n), each forming the transmitted symbol, $s_{m,n}$, and totally M symbol vectors, $\mathbf{s}_p$, with $p = 1, \ldots, M$, $M = 4^k$, $k \in \mathbf{Z}^+$ and $\{m, n\} \in [1, \ldots, \sqrt{M}]$. The constellation structure is illustrated in Fig. 1 for $M = 16$, where the SQAM symbols are depicted as the squared ones. The SQAM symbols are placed on the corners of squares with sides equal to $2d$. Thus, the euclidean distance between any two adjacent symbols is $2d$, with d depending on the modulation order, M, and changes accordingly, so that the total average constellation's energy remains equal to E_{av}.

In θ-QAM family constellations, the distances between symbols are as follows

$$\mathcal{D}\left(s_{m,n},\ s_{m,n+1}\right) = 2d, \quad n = 1, \ldots, \sqrt{M}-1$$
$$\mathcal{D}\left(s_{m,n},\ s_{m+1,n}\right) = 2d, \quad m = 1, \ldots, \sqrt{M}-1$$

but the symbols do not lie at the edges of a square but on the corners of isosceles triangles; the constellation's symmetry with respect to the point $(0, 0)$ is maintained. The uneuqal angle, θ, of those isosceles triangles affects the euclidean distance of the non-adjacent symbols as shown in Fig. 1, where θ-QAM symbols are depicted for various values of the angle, θ (the TQAM constellation's symbols are shown as triangles). Practically, this angle affects the value of the total average energy per symbol, E_{av}. For example, for the case of TQAM, which belongs to the θ-QAM family, when $\theta = \pi/3$, it is known that the average symbol energy is less than that of SQAM, finally leading in lower SER, since for the same E_{av}, the minimum euclidean distance increases [13].

The coordinates of the symbols, $s_{m,n}$, of an 16-ary θ-QAM constellation, can be derived after some simple trigonometric calculations as shown in Fig. 1 and can be generalized to the case of an M-ary θ-QAM, since any higher order can be directly constructed by placing lower order constellatons adjacently. Finally, the coordinates x_m, y_m, of the symbol, $s_{m,n}$, of a M-ary θ-QAM constellation are given by

$$(x_m,\ y_n) = \left(\left[2(n\text{-}1)\text{+}1\text{-}\sqrt{M}\right] d\text{+}\left[2\mathrm{mod}(m,2)\text{-}1\right]\frac{a}{2},\right. \quad (1)$$
$$\left. \text{-}\left[2(m\text{-}1)\text{+}1\text{-}\sqrt{M}\right]\frac{b}{2}\right), \quad (2)$$

for $m = 1, \ldots, \sqrt{M}$, $\quad n = 1, \ldots, \sqrt{M}$. Moreover, $\mathrm{mod}(\cdot)$ denotes the modulus after division, $2d$ is the euclidean distance between the adjacent signal points, $a = 2d\cos\theta$ and $b = 2d\sin\theta$. Note, that from a theoretical point of view, the angle, θ, may take values inside the interval $(0,\ \pi)$, but due to symmetry only the angles lower than $\pi/2$ are considered.

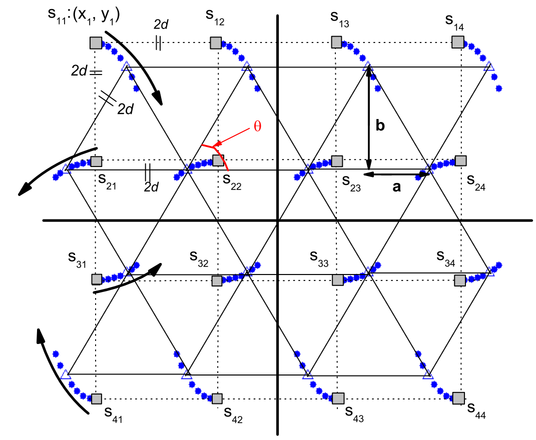

Fig. 1. The structure of the θ-QAM constellation. SQAM symbols are depicted as squares, while the TQAM ones as triangles.

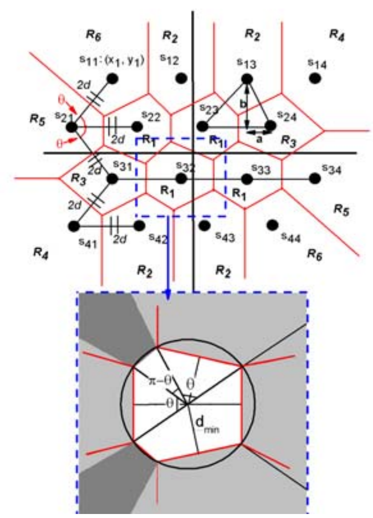

Fig. 2. The decision boundaries of a θ-QAM constellation.

A. Average Energy per Symbol

A random θ-QAM constellation is shown in Fig. 2. The symbol energy is, $E_S = x_m^2 + y_n^2$, and using (1) and (2) it can be expressed as

$$E_{av}=\frac{2}{M}\sum_{i=0}^{\sqrt{M}-1}\sum_{j=0}^{\frac{\sqrt{M}}{2}-1}\left\{\left[\left(2i\text{+}1\text{-}\sqrt{M}\right)d\text{-}\frac{a}{2}\right]^2+\left[(2j\text{+}1)\frac{b}{2}\right]^2\right\} \quad (3)$$

After some simplifications (3) reduces to

$$E_{av} = [3M + (4-M)\cos 2\theta]d^2/6, \quad (4)$$

which can be solved with respect to d, as

$$d = \sqrt{6E_{av}/[3M + (4-M)\cos 2\theta]}. \quad (5)$$

B. Decision Boundaries and SER Evaluation in AWGN Channels

The decision boundaries of a random θ-QAM constellation are shown in Fig. 2 (red lines). As it is shown, there are six different types of regions formed by the decision boundaries. These regions are denoted as R_1, R_2, …, R_6 and they will be thus referred hereafter. It must be noted that the shapes of these regions are the same regardless of the modulation order; only the number of repetitions changes. For a general M-ary θ-QAM constellation, it can be easily shown by observation, that the repetition number, N_{Ri} for the i-th region is given as following: $N_{R_1} = (\sqrt{M} - 2)^2$, $N_{R_2} = 2(\sqrt{M} - 2)$, $N_{R_3} = \sqrt{M} - 2$, $N_{R_4} = 2$, $N_{R_5} = \sqrt{M} - 2$, $N_{R_6} = 2$. The SER of this constellation can be evaluated by calculating for each transmitted symbol vector, $\mathbf{s_p}$, the probability that the received vector, $\mathbf{r}$, does not lie inside the corresponding decision region, R_p, and then taking the average of these probabilities, i.e.,

$$P_e = \frac{1}{M}\sum_{p=1}^{M}\left(\int_{\overline{R_p}} f\left(\mathbf{r}|\mathbf{s_p}\right)dr\right) \tag{6}$$

where $f\left(\mathbf{r}|s_p\right)$ is the probability density function (pdf) of the received signal vector $\mathbf{r}$ conditioned on the transmitted signal, $\mathbf{s_p}$, and $\overline{R_p}$ is the complementary of the decision region R_p. Next, using cylindrical coordinates we will calculate the error probability for each region.

After dividing the region $\overline{R_1}$, into sectors (see Fig.2), the total probability of error for this region is

$$P_{R_1}{=}4\int_{\frac{\pi-\theta}{2}}^{\frac{\pi+\theta}{2}}\int_{\frac{d}{\sin\phi}}^{\infty} g(r)rdrd\phi{+}2\int_{\theta}^{\pi-\theta}\int_{\frac{d\sin\theta}{\sin\phi\cos\frac{\theta}{2}}}^{\infty} g(r)rdrd\phi \tag{7}$$

where $g(r) = \frac{1}{\pi N_0}e^{-\frac{1}{N_0}r^2}$, with N_0 being the one-side power spectral density of the AWGN. Similarly, the probability of error for the other regions can be evaluated as in (8). After some trivial mathematical manipulations, the SER can be written as in (9) at the top of the next page, where $\gamma = E_{av}/N_0$ is the average SNR, $\delta = d/\sqrt{Eav}$, $\csc(\cdot) = 1/\sin(\cdot)$ and $c_1 = 1/(2\pi M)$, $c_2 = 4(\sqrt{M} - 2)(\sqrt{M} - 1)$, $c_3 = \sqrt{2}(\sqrt{M}-1)$, $c_4 = 5(\sqrt{M}-2)+6$, $c_5 = 3(\sqrt{M}-2)+2$ and $c_6 = \sqrt{M}$.

It can be easily verified that for the case of $\theta = \pi/2$, (9) reduces to known SER expressions of SQAM [20], while for $\theta = \pi/3$ reduces to the SER of TQAM, which to the best of the authors' knowledge, has not been given in an analytical form in the past.

III. Optimum θ-QAM Constellations

As mentioned in the introduction, the optimum QAM scheme in terms of transmission efficiency is generally unknown. In this section, we find the optimum θ-QAM constellations for any value of the SNR and any order of the modulation, using the exact analytical expression for the SER derived in the previous section. Using this expression for constant values of the SNR, $\overline{\gamma}$, and M, the optimum angle (which minimizes the SER) can be calculated by minimizing $P_S(\overline{\gamma}, \theta, M)$ with respect to θ, applying well known numerical optimization methods, such as the effective and computationally compact *Nelder-Mead* method, which is included in many mathematical software packages [21].

As an example, the optimum angles for $M = 16$ and $M = 64$ are presented in Fig. 3, where we observe that as M increases, the variability of the optimum angle becomes larger. For example, when $M = 16$, θ varies from $\theta = 57(\pi/180)$ to $\theta = 63(\pi/180)$, while for $M = 64$, θ varies from $\theta = 38(\pi/180)$ to $\theta = 60(\pi/180)$. Moreover, it can be seen that for high SNRs the optimum angle is almost always $\theta = \pi/3$, regardless of the modulation order.

This occurs because for $\theta = \pi/3$ (TQAM), the regions of decision of the inner part of the constellation are regular hexagons, which provide optimum coverage of the area (that is less energy for the same minimum distance between adjacent symbols). Thus, it would be expected that $\theta = \pi/3$ should always be the optimum angle. This does not happen though, because the regions of decision at the edges of the constellation are not hexagons and they do not include the maximum possible part of the Gaussian probability distribution for $\theta = \pi/3$, when the SNR is low.

In Fig. 4, the SER is plotted against the angle, θ, for specific SNRs. For low error probabilities, the effect of the optimum angle, θ, on the SER is obvious; for $M = 16$ and SNR=20 dB, the SER varies between $P_e = 4 \times 10^{-6}$ and $P_e = 10^{-3}$. On the other hand, for higher error probabilities, the effect of θ on the SER is negligible. In other words, there is no difference in the SER whether SQAM, or TQAM or any other θ-QAM constellations is applied. If we take also into account the fact that in the high SNR region, the optimum angle approaches $\theta = \pi/3$, TQAM seems to be a sufficient compromise for achieving approximately the minimum SER over the whole SNR region. However, this claim is valid only if system's complexity is ignored (e.g. the demodulation of TQAM is more complex than that of SQAM). At this point we should note that a practical scheme with a variable angle, θ, according to channel conditions, optimizing the error performance, would require a feedback chanell. This scheme, would be similar to an adaptive modulation scheme, with the difference that the angle, θ, would be an extra varying parameter.

IV. Bit Stream Mapping and BER Evaluation of θ-QAM Constellations

A. Bit Stream Mapping

In θ-QAM constellations perfect Gray coding is not possible, since there are more than four adjacent neighbors around almost every signal point. A bit stream mapping for the special case of 16-TQAM has been proposed in [13]. This is the optimum when $\theta > \pi/3$, because the adjacent symbols located at a distance $2d$, which is in that case the minimum distance between adjacent symbols, differ only in one bit . The adjacent symbols that differ in more than one bits are located at a distance longer than $2d$, minimizing the probability of having a symbol error of that kind. In general, the most probable symbol errors between adjacent symbols in θ-QAM constellations, are those that occur between symbols that are located at a distance $2d$.

B. Gray Code Penalty

Since perfect Gray coding is not possible for θ-QAM, a Gray Code Penalty (GCP) will occur. Given the GCP, G_p,

$$
\begin{aligned}
P_{R_2} &= \int_{\frac{\pi-\theta}{2}}^{\frac{\pi+\theta}{2}} \int_{\frac{d}{\sin\phi}}^{\infty} g(r) r dr d\phi + \int_{\theta}^{\pi-\theta} \int_{\frac{d\sin\theta}{\sin\phi\cos\frac{\theta}{2}}}^{\infty} g(r) r dr d\phi + 2\int_{\frac{\pi-\theta}{2}}^{\pi} \int_{\frac{d}{\sin\phi}}^{\infty} g(r) r dr d\phi \\
P_{R_3} &= \int_{\frac{\pi-\theta}{2}}^{\frac{\pi+\theta}{2}} \int_{\frac{d}{\sin\phi}}^{\infty} g(r) r dr d\phi + 2\int_{\theta}^{\pi-\theta} \int_{\frac{d\sin\theta}{\sin\phi\cos\frac{\theta}{2}}}^{\infty} g(r) r dr d\phi + 2\int_{\frac{\pi-\theta}{2}}^{\frac{\pi+2\theta}{2}} \int_{\frac{d}{\sin\phi}}^{\infty} g(r) r dr d\phi \\
P_{R_4} &= \int_{\frac{\pi-\theta}{2}}^{\pi} \int_{\frac{d}{\sin\phi}}^{\infty} g(r) r dr d\phi + \int_{\frac{\pi-\theta}{2}}^{\frac{\pi+2\theta}{2}} \int_{\frac{d}{\sin\phi}}^{\infty} g(r) r dr d\phi + \int_{2\theta}^{\pi} \int_{\frac{2d\sin\theta}{\sin\phi}}^{\infty} g(r) r dr d\phi \\
P_{R_5} &= \int_{\frac{\pi-\theta}{2}}^{\frac{\pi+\theta}{2}} \int_{\frac{d}{\sin\phi}}^{\infty} g(r) r dr d\phi + \int_{\frac{\pi-\theta}{2}}^{\pi} \int_{\frac{d}{\sin\phi}}^{\infty} g(r) r dr d\phi + \int_{\frac{\pi-\theta}{2}}^{\frac{\pi+2\theta}{2}} \int_{\frac{d}{\sin\phi}}^{\infty} g(r) r dr d\phi + \int_{2\theta}^{\pi} \int_{\frac{2d\sin\theta}{\sin\phi}}^{\infty} g(r) r dr d\phi \\
P_{R_6} &= \int_{\theta}^{\pi-\theta} \int_{\frac{d\sin\theta}{\sin\phi}}^{\infty} g(r) r dr d\phi + 2\int_{\frac{\pi-\theta}{2}}^{\pi} \int_{\frac{d}{\sin\phi}}^{\infty} g(r) r dr d\phi
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
P_S(\gamma,\theta,M) &= \frac{1}{M}\left(N_{R_1}P_{R_1} + N_{R_2}P_{R_2} + N_{R_3}P_{R_3} + N_{R_4}P_{R_4} + N_{R_5}P_{R_5} + N_{R_6}P_{R_6}\right) \\
&= c_1 \Bigg\{ c_2 \int_{\frac{\pi-\theta}{2}}^{\frac{\pi+\theta}{2}} \exp\left(-\gamma\delta^2\csc^2\phi\right) d\phi + c_3^2 \int_{\theta}^{\pi-\theta} \exp\left(-\gamma\delta^2\csc^2\phi\sec^2\frac{\theta}{2}\sin^2\theta\right) d\phi \\
&\quad + c_4 \int_{\frac{\pi-\theta}{2}}^{\pi} \exp\left(-\gamma\delta^2\csc^2\phi\right) d\phi + c_5 \int_{\frac{\pi-\theta}{2}}^{\frac{\pi+2\theta}{2}} \exp\left(-\gamma\delta^2\csc^2\phi\right) d\phi \\
&\quad + c_6 \int_{2\theta}^{\pi} \exp\left(-4\,\gamma\delta^2\csc^2\phi\sin^2\theta\right) d\phi \Bigg\}
\end{aligned} \tag{9}
$$

and the SER, P_S, the BER can be evaluated as [14]

$$P_b = \frac{G_p}{\log_2 M} P_S. \tag{10}$$

The GCP for SQAM and TQAM is [14]

$$G_p = \frac{1}{M}\sum_{i=1}^{M} G_p^{S_i} = \frac{1}{M}\sum_{i=1}^{M} \frac{\sum_{j=1}^{M(S_i)} B_d(S_i,S_j)}{N(S_i)} \tag{11}$$

where M is the constellation size, S_i denotes ith symbol, $G_p^{S_i}$ is the Gray Code Penalty of the symbol S_i, $N(S_i)$ is the number of the nearest neighbors of S_i and $B_d(S_i,S_j)$ is the number of bit difference between S_i and the nearest neighboring symbol of S_i, S_j. This is possible because, in SQAM and TQAM, the distance between every two adjacent symbols is always equal to $2d$ (see Fig. 1).

On the other hand, in a random θ-QAM constellation, it is not possible to use (11) for the evaluation of the GCP, because the adjacent symbols do not contribute in the same way to GCP, due to the fact that they are placed in different distances from the signal point. Next, we propose an alternative method, which can be efficiently used for approximating the BER of the random θ-QAM scheme.

C. BER Approximation

When evaluating the SER in each region (see Figure 2), we formed two integrals for two different types of areas of integration. For the BER evaluation at high SNR values, we can assume that the errors occur very close to the limits of each region. Thus, we can multiply – as an approximation – each of these integrals with the corresponding number of different bits between the adjacent regions. If we follow this approach with all types of regions, the BER approximation will be given by (9), but now $c_1 = 1/(2\pi M \log_2 M)$, $c_3 = 2(\sqrt{M}-1)$ and $c_6 = 2\sqrt{M}$, while c_2, c_4 and c_5 remain the same. In Fig.5, the BER of θ-QAM is plotted using the proposed method. As it can be seen, the approximation is highly accurate even for the medium and low SNRs.

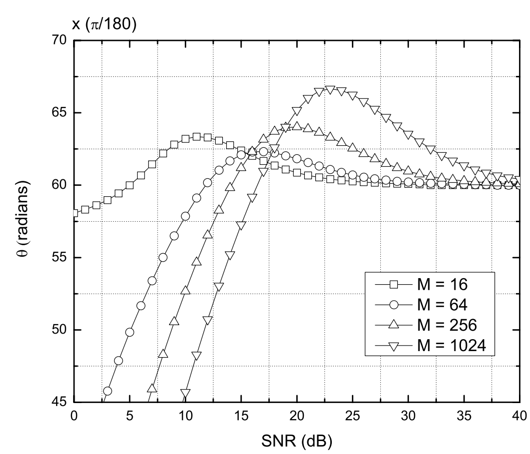

Fig. 3. The optimum angles θ, for various values of the SNR and the modulation order M in AWGN.

The most interesting result, regarding the BER is the fact that *TQAM does not always achieve a lower error probability than SQAM as in the case of the SER*. On the contrary, in some cases SQAM achieves the minimum BER among all the θ-QAM constellations. As previously discussed, the reason is

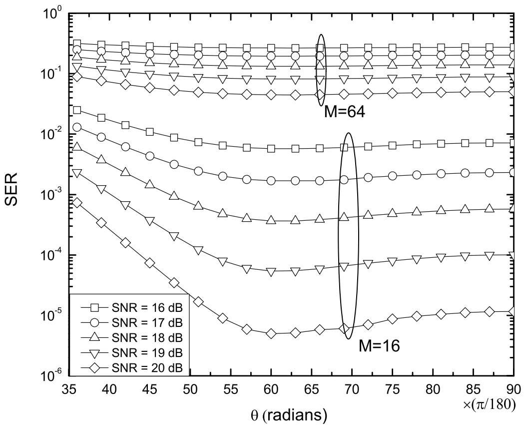

Fig. 4. The SER of θ-QAM versus the angle, θ in AWGN.

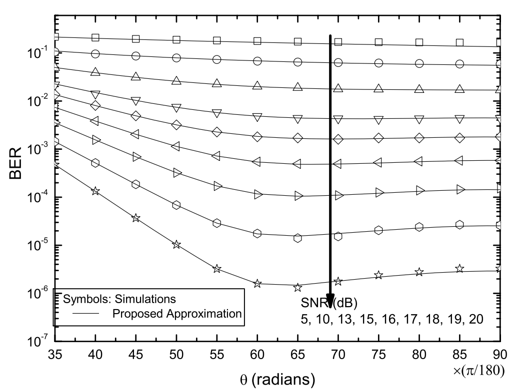

Fig. 5. The approximated BER of θ-QAM for $M = 16$ in AWGN.

that for SQAM perfect Gray coding is possible, while for all the other θ-QAM constellations, it is not. It is evident that for the high SNR region (SNR >15 dB) the minimum BER corresponds to an angle $\theta \approx 65^o$, while for the medium and low SNR region (SNR <15 dB), the optimum constellation is the SQAM. *From a practical aspect, TQAM is not the most appropriate constellation in terms of BER.*

V. Performance over Nakagami-m Fading Channels

A. SER Evaluation of θ-QAM

In the case of Nakagami-m fading, the SNR, γ, of the received signal will be a gamma distributed random variable, i.e., [22]

$$f_\gamma(\gamma) = \frac{m^m \gamma^{m-1}}{\overline{\gamma}^m \Gamma(m)} \exp\left(-\frac{m\gamma}{\overline{\gamma}}\right) \tag{12}$$

where $\overline{\gamma}$ is the average SNR, m is the Nakagami-m fading parameter, which ranges from 0.5 to ∞ and $\Gamma(\cdot)$ denotes the complete Gamma function.

The SER evaluation of θ-QAM in Nakagami-m fading, with integer values of m can be evaluated by averaging (9) over the (12), which reduces to (13), after solving the inner integral and using [19, 3.351 (3)].

Now, the integrals involved in (13) have the following form

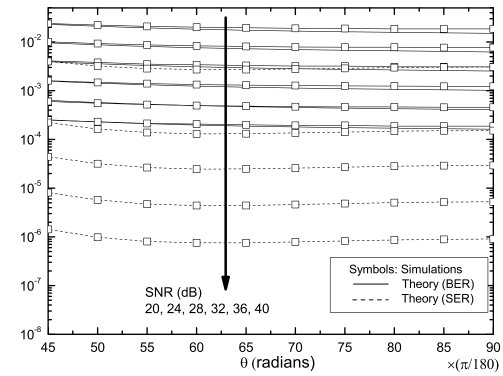

Fig. 6. The SER ($m = 3$) and the approximated BER ($m = 1$) of θ-QAM for $M = 16$ in Nakagami-m fading.

$$I_1 = \int_{l_1}^{l_2} \left(\frac{m}{m + A\csc^2 x}\right)^m dx \tag{14}$$

which can be written as

$$\begin{aligned} I_1 &= \int_{l_1}^{l_2} \left(\frac{m\sin^2 x}{m\sin^2 x + A}\right)^m dx \overset{y=2x}{=} \\ &= \frac{1}{2}\int_{2l_1}^{2l_2} \left(1 + \frac{\alpha}{\beta + \cos y}\right)^m dy \end{aligned} \tag{15}$$

with $\alpha = 2A/m$ and $\beta = -1 - 2A/m$. By using the binomial expansion of the term $\left(1 + \frac{\alpha}{\beta+\cos y}\right)^m$, and after exchanging the order of summation and integration, we get

$$I_1 = \frac{1}{2}\sum_{k=0}^{m} \binom{m}{k} \int_{2l_1}^{2l_2} \frac{\alpha^k}{(\beta + \cos y)^k} dy \tag{16}$$

The integral in (16) can be evaluated in closed-form in terms of elementary functions using the recursive solution of the indefinite integral [19, 2.554 (3)].

$$\begin{aligned} \int \frac{\alpha^k}{(\beta + \cos y)^k} dy &= \frac{\alpha^k}{(k-1)(\beta^2 - 1)} \left\{ \frac{\sin y}{(\beta + \cos x)^{k-1}} \right. \\ &\left. - \int \frac{(k-1)\beta - (k-2)\cos y}{(\beta + \cos y)^{k-1}} dy \right. \end{aligned} \tag{17}$$

and the integral [19, 2.554 (1)].

$$\begin{aligned} \int \frac{A + B\cos y}{(\beta + \cos y)^k} dy &= \frac{1}{(k-1)(\beta^2 - 1)} \left\{ \frac{(\beta B - A)\sin y}{(\beta + \cos x)^{k-1}} \right. \\ &\left. - \int \frac{(A\beta - B)(k-1) + (k-2)(\beta B - A)\cos y}{(\beta + \cos y)^{k-1}} dy \right. \end{aligned} \tag{18}$$

which for k=1 reduces to

$$\begin{aligned} \int \frac{A + B\cos y}{\beta + \cos y} dy &= By + (A - \beta B) \\ &\times \begin{cases} \frac{2}{\sqrt{\beta^2-1}} \arctan \frac{\sqrt{\beta^2-1}\tan\frac{y}{2}}{\beta+1}, & [\beta^2 > 1] \\ -\frac{1}{\sqrt{1-\beta^2}} \ln \frac{\sqrt{1-\beta^2}\tan\frac{y}{2} + \beta + 1}{\sqrt{1-\beta^2}\tan\frac{y}{2} - \beta - 1}, & [\beta^2 < 1] \end{cases} \end{aligned} \tag{19}$$

$$P_{S_N} = c_1 \left\{ c_2 \int_{\frac{\pi-\theta}{2}}^{\frac{\pi+\theta}{2}} \left(\frac{m}{m + \delta^2 \csc^2 \phi \overline{\gamma}} \right)^m d\phi + c_3^2 \int_{\theta}^{\pi-\theta} \left(\frac{m}{m + \delta^2 \csc^2 \sec^2 \frac{\theta}{2} \sin^2 \theta \phi \overline{\gamma}} \right)^m d\phi \right.$$
$$+ c_4 \left(\int_{\frac{\pi-\theta}{2}}^{\pi} \left(\frac{m}{m + \delta^2 \csc^2 \phi \overline{\gamma}} \right)^m d\phi \right) + c_5 \int_{\frac{\pi-\theta}{2}}^{\frac{\pi+2\theta}{2}} \left(\frac{m}{m + \delta^2 \csc^2 \phi \overline{\gamma}} \right)^m d\phi$$
$$\left. + c_6 \left(\int_{2\theta}^{\pi} \left(\frac{m}{m + 4\delta^2 \csc^2 \phi \sin^2 \theta \overline{\gamma}} \right)^m d\phi \right) \right\} \quad (13)$$

1) Special case: Rayleigh fading ($m = 1$): For the important case of Rayleigh fading, (16) reduces to [19, 2.554 (3)]

$$I_1 = \int_{l_1}^{l_2} \frac{1}{1{+}A \csc^2 x} dx = \left[l_2 - \sqrt{\frac{A}{1{+}A}} \operatorname{atan} \left(\frac{\sqrt{1{+}A} \tan l_2}{\sqrt{A}} \right) \right] - \left[l_1 - \sqrt{\frac{A}{1{+}A}} \operatorname{atan} \left(\frac{\sqrt{1{+}A} \tan l_1}{\sqrt{A}} \right) \right], \; l_1, \; l_2 \neq \frac{\pi}{2} \quad (20)$$

By applying (20) in (13) we get a closed-form solution for the SER of θ-QAM in Rayleigh fading, in terms of elementary functions.

B. BER Approximation

Following the same procedure as that of the case of AWGN, the BER of θ-QAM in Nakagami-m fading can be approximated by (13) where now $c_1 = 1/(2\pi M \log_2 M)$, $c_3 = 2(\sqrt{M} - 1)$ and $c_6 = 2\sqrt{M}$. In Fig. 6, the BER is depicted against the angle, θ, assuming Rayleigh fading. It is interesting to note that *the minimum BER is achieved by the SQAM*. For example, when $\overline{\gamma} = 40$ dB, TQAM results in a BER of 2×10^{-4}, while SQAM in a BER of 1.5×10^{-4}. In general, one can observe that *for the case of fading channels the choice of any of the θ-QAM constellations should be based on the demodulation complexity*. For example, considering an application over fading channels, there should be no dilemma between choosing TQAM or SQAM as the modulation scheme.

VI. Conclusions

A parametric family of quadrature modulation was studied, called θ-QAM, which includes other known quadrature modulations, such as SQAM and TQAM, as special cases. Exact analytical expressions for the SER and highly accurate approximations for the BER in AWGN were presented. Moreover, exact closed-form expressions for the SER in Nakagami-m fading channels were derived. By evaluating the exact SER and the approximated BER of θ-QAM, we found the optimum angles, which minimize the SER or the BER for a specific signal-to-noise ratio (SNR) and modulation order, M. This serves as an indicator for the appropriate constellation with respect to channel conditions and SER or BER requirements. It was shown that, TQAM achieves approximately the minimum SER over the whole range of SNRs and that it is an efficient solution when the choice of the modulation is based on the SER minimization. On the other hand, the minimum BER is achieved for an angle different from $\theta = 60^o$ (TQAM). Furthermore, it was shown that in fading channels, SQAM is the optimum constellation in terms of BER.

References

[1] R. W. Lucky and J. C. Hancock, "On the optimum performance of N-ary systems having two degrees of freedom," *IRE Trans. Commun. Syst.*, vol. CS-10, pp. 185-192, June 1962.

[2] J. Salz, J. R. Sheehan, and D. J. Paris, "Data transmission by combined AM and PM," *Bell Syst. Tech. J.*, vol. 50, Sept. 1971.

[3] K. Kawai, S. Shintani, and H. Yanagidaira, "Optimum combination of amplitude and phase modulation scheme," in *1972 Int. Conf.: Commun. Rec.*, Philadelphia, PA

[4] R. W. Lucky, J. Salz, and E. J. Weldon, Jr., *Principles of Data Communication*. New York: McGraw-Hill, 1968.

[5] C. Campopiano and B. Glazer, "A coherent digital amplitude and phase modulation scheme," *IRE Trans. Commun.*, vol. 10, pp. 90-95, Dec. 1962.

[6] A. J. Goldsmith and S.-G. Chua, "Variable-rate variable-power M-QAM for fading channel," *IEEE Trans. Commun.*, vol. 45, no. 10, pp. 1218-1230, Oct. 1997.

[7] W. T. Webb and R. Steele, "Variable rate QAM for mobile radio," *IEEE Trans. Commun.*, vol. 43, pp. 2223-2230, July 1995.

[8] X. Qiu and K. Chawla, "On the performance of adaptive modulation in cellular systems," *IEEE Trans. Commun.*, vol. 47, no. 6, pp. 884-895, June 1999

[9] A. Conti, M. Z. Win, and M. Chiani, "Slow adaptive M-QAM with diversity in fast fading and shadowing," *IEEE Trans. Commun.*, vol. 55, no. 5, pp. 895-905, May 2007.

[10] G. Foschini, R. Gitlin, and S. Weinstein, "Optimization of two dimensional signal constellations in the presence of Gaussian noise," *IEEE Trans. Commun.*, vol. 22, pp. 28-38, Jan. 1974.

[11] M. Kifle and M. Vanderaar, "Bounds and simulation results of 32-ary and 64-ary quadrature amplitude modulation for broadband-ISDN via satellite," NASA Technical Memorandum 106484, Feb. 1994.

[12] A. Conti, D. Panchenko, S. Sidenko, and V. Tralli, "Log-concavity property of the error probability with application to local bounds for wireless communications," *IEEE Trans. Inf. Theory*, vol. 55, no. 6, pp. 2766-2775, June 2009.

[13] S.-J. Park, "Triangular quadrature amplitude modulation," *IEEE Commun. Lett.*, vol. 11, no. 4, pp. 292-294, Apr. 2007.

[14] S.-J. Park, "Bit mapping of triangular quadrature amplitude modulation," *18th Annual IEEE International Symp. Personal, Indoor Mobile Radio Commun. (PIMRC'07)*, 1-4244-1144-0/07.

[15] J. Lu, K. B. Letaief, J. C.-I. Chuang, and M. L. Liou, *M*-PSK and *M*-QAM BER computation using signal-space concepts," *IEEE Trans. Commun.*, vol. 47, no. 2, pp. 181-184, Feb. 1999.

[16] L.-L. Yang and L. Hanzo, "A recursive algorithm for the error probability evaluation of *M*-QAM," *IEEE Commun. Lett.*, vol. 4, no. 10, pp. 304-306, Oct. 2000.

[17] K. Cho and D. Yoon, "On the general BER expression of one- and two-dimensional amplitude modulations," *IEEE Trans. Commun.*, vol. 50, no. 7, pp. 1074-1080, July 2002.

[18] A. Conti, M. Z. Win, and M. Chiani, "Invertible bounds for M-QAM in Rayleigh fading," *IEEE Trans. Wireless Commun.*, vol. 4, no. 5, pp. 1994-2000, Sept. 2005

[19] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 6th ed. New York: Academic, 2000.

[20] J. G. Proakis, *Digital Communications*, 4th ed. New York: McGraw-Hill, 2001.

[21] J. A. Nelder and R. Mead, "A simplex method for function minimization," *Computer J.*, vol. 7, pp. 308-313, 1965.

[22] M. K. Simon and M. Alouini, *Digital Communication over Fading Channels*, 2nd ed. John Wiley, 2005.